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Abstract

In this study, we examine the validity of the collective rationality condition in a two-stage collective choice procedure with the process of nomination. We first show that even the minimal requirement of collective rationality implies that the opinions of nominators, which are expressed in the first stage of collective choice, must be irrelevant to the second stage of collective choice, in which the preferences of voters are expressed. This result has a difficulty in that the minimal collective rationality condition must separate the first stage from the second stage of collective choice. Alternatively, we propose *nominee-efficient* collective choice, in which the opinions of nominators influence the second stage of collective choice. Then, a new concept of conditional (on the opinions of nominators) rationalizability and its axiomatic foundation are provided. Furthermore, we characterize the process of nomination under which nominee-efficient collective choice is compatible with Arrow's well-known axioms.

JEL classification: D71, D72

Keywords: nomination, nominee-efficiency, collective rationality, conditional rationalizability, collective choice

1 Introduction

In this study, we examine the validity of the collective rationality condition within a broader conceptual framework for collective choice problems. One of the research directions in collective choice theory is how to relax the collective rationality condition imposed by Arrow (1963)—the condition that all social rankings are orderings—to escape from his well-known impossibility theorem.[1] In a classical work, Sen (1969, 1970) characterizes the Pareto extension rule by weakening the transitivity of social rankings to their quasi-transitivity. Weymark (1984) provides a characterization of the Pareto rule by dropping the

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[1]See Gibbard (2014), Mas-Colell and Sonnenschein (1972), and Brown (1975) for negative results obtained by weakening the collective rationality condition.

completeness of social rankings. Bossert and Suzumura (2008) provide a characterization of collective choice rules satisfying Suzumura-consistency (Suzumura 1976), which is logically between acyclicity and transitivity but is independent of quasi-transitivity.[2]

All these studies weaken the collective rationality condition, while maintaining the minimal condition of collective rationality. The minimal condition of collective rationality guarantees the existence of the social rankings associated with the preferences of individuals and requires that according to the social rankings, the *greatest elements* are collectively chosen from the set of feasible alternatives, where the social rankings are binary relations over the set of all alternatives.

In this study, we point out that even the minimal requirement of collective rationality may be too demanding within the extended collective choice framework recently proposed by Iwata (2016). We illustrate an extended framework that consists of a two-stage collective choice procedure. Suppose that a university invites applications for an academic position. Every applicant needs at least three recommenders for his or her application. If the number of applicants is very large, referees might evaluate their works. In the first stage of collective choice, the list of final candidates, called *nominees*, is determined by the opinions of recommenders and referees, called *nominators*. Their opinions are expressed in one of three ways: they express their positive, negative, or neutral opinions about which potential applicants become eligible as nominees.[3] In the second stage of collective choice, the university makes a collective choice from the list of nominees according to the preferences of its members, called *voters*. Thus, the second stage of collective choice is based on a standard Arrovian collective choice model.[4] Iwata (2016) shows that Arrow's impossibility theorem is resolved within the extended collective choice framework when there exists at least one nominating voter.

As pointed out by Iwata (2016), a choice consistency condition, which is a counterpart of Arrow's (1959) choice axiom imposed on the two-stage collective choice procedures explained above and can be interpreted as a collective rationality condition, implies that the final choice does not change if nominated alternatives (i.e., nominees) remain the same when the opinions of nominators change. We refer to this property as "nominee invariance." Nominee invariance suggests that the opinions of nominators are irrelevant to the second stage of collective choice, which implies that the collective rationality condition necessarily separates the first stage from the second stage of collective choice. In this study, we provide a result that formalizes this statement. More precisely, we show that even the minimal requirement of collective rationality implies nominee invariance. To show this result, we need only the property that the social

[2]See Bossert and Suzumura (2010) and Suzumura (1976, 1983) for further discussions and the implications of Suzumura-consistency.

[3]See Ju (2010), Kasher and Rubinstein (1997), and Samet and Schmeidler (2003) for opinion aggregation problems related to the process of nomination explained here.

[4]Technically, the second stage of collective choice has a similar structure to the model developed by Denicolò (1985, 1993).

rankings are (reflexive) binary relations over the set of all alternatives, and do not need any other properties such as completeness or acyclicity.

However, we argue that nominee invariance is a too demanding requirement because the information about the opinions of nominators cannot be used in the second stage of collective choice after determining nominees in the first stage of collective choice. To understand this argument, consider the example illustrated above. The information in recommendation letters and referees' reports is often used not only to determine the list of nominees (e.g., as a nomination qualification), but also to make the final choice from the list of nominees (e.g., as a reference to the final choice). That is, any information about the recommenders or referees (e.g., who recommends an applicant or how referees evaluate the applicants) could play an important role in making a decision in both the first stage and the second stage of collective choice. This indicates that even the minimal collective rationality condition is too demanding in two-stage collective choice procedures because it implies nominee invariance.

Alternatively, in this study, we propose a new concept of conditional or bounded collective rationality under which the opinions of nominators influence the second stage of collective choice. *Nominee-efficient collective choice* is based on the idea that the society (e.g., the university) makes a collective choice from the set of nominated alternatives (e.g., nominees) that are *efficient* in terms of the opinions of nominators (e.g., recommenders and referees), instead of the set of all nominated alternatives. The concept of *nominee-efficient rationalizability* is more modest than any standard concept of rationalizability, which requires that according to a social ranking, the finally chosen alternatives weakly dominate only nominated alternatives that are efficient in terms of the opinions of nominators, instead of all the nominated alternatives. In this study, we provide an axiomatic foundation for the concept of nominee-efficient rationalizability.

Our main contribution is to characterize the process of nomination under which the property of nominee-efficient rationalizability is compatible with Arrow's well-known axioms. Since we impose no restriction on the process of nomination, our result completely characterizes when Arrow's impossibility theorem is resolved under nominee-efficient collective choice. Thus, we identify a crucial boundary between the possibility and impossibility of two-stage collective choice procedures.

The remaining sections are organized as follows. Section 2 introduces the model proposed by Iwata (2016). Section 3 discusses the minimal concept of rationalizability and its implications. Section 4 proposes nominee-efficient collective choice and provides a characterization of nominee-efficient rationalizability. Section 5 characterizes the process of nomination under which nominee-efficient collective choice is compatible with Arrow's axioms. Section 6 concludes. Some of the proofs of the results are presented in the appendix.

2 The Model

Let $\mathbb{N}$ be the set of all natural numbers. Let $X = \{x_1, x_2, \ldots, x_s\}$ with $\#X = s \geq 3$ be a finite set of potentially feasible alternatives, where $\#$ denotes the cardinality of a set. When we consider the situation in which the numbering of alternatives is not always needed, we may replace $X = \{x_1, x_2, \ldots, x_s\}$ with $X = \{x, y, \ldots, z\}$. Let A denote a non-empty subset of X. Let $\mathscr{X}$ be the set of all non-empty subsets of X. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ with $n \geq 2$ be a finite set of individuals. Individuals are *nominators*, *voters*, or both. Let $\mathcal{M}$ be the set of nominators and $\mathcal{V}$ be the set of voters, and $\mathcal{N} = \mathcal{M} \cup \mathcal{V}$, where $\#\mathcal{M} = m \geq 2$, $\#\mathcal{V} = v \geq 2$, and $\mathcal{M} \cap \mathcal{V}$ is not always the empty set.

2.1 The first stage of collective choice

We introduce a two-stage collective choice procedure in which some alternatives are first nominated and then the final choice is made from the set of nominated alternatives. First, we describe the first stage of the collective choice procedure. Each nominator $i \in \mathcal{M}$ expresses an *opinion* over X. Nominator i's opinion is a view that he or she has concerning which alternative should be eligible for collective decision-making. Such an opinion is represented by a $1 \times s$ row vector J_i consisting of 1, 0, or -1. Let J_{ik} denote the kth component of J_i. The interpretation of $J_{ik} = 1$ (respectively, $J_{ik} = 0$ or $J_{ik} = -1$) is that nominator i has a positive (respectively, neutral or negative) view of alternative x_k. An *opinion profile* J is an $m \times s$ matrix consisting of m row vectors $J_1, \ldots, J_m$. Let $\mathscr{J}$ be the set of all opinion profiles.

In the first stage of collective choice, the opinions of nominators are aggregated to identify which alternatives are nominated. The process of nomination is described as a *nomination rule*, $f : \mathscr{J} \to \mathscr{X} \cup \{\emptyset\}$, which assigns a non-empty subset of X or the empty set to each opinion profile. For all $J \in \mathscr{J}$, let $f(J)$ denote the set of *nominated alternatives* or *nominees*.[5] That is, $x_k \in f(J)$ if and only if x_k is a nominated alternative. Since we do not restrict the nomination rules, the collection of the sets of nominated alternatives is arbitrarily decided as a set of the subsets of $\mathscr{X} \cup \{\emptyset\}$ according to a given nomination rule. Therefore, the set of nominated alternatives may have no structure such that any binary choice is in the collection of the sets of nominated alternatives or all alternatives can be simultaneously nominated.

2.2 The second stage of collective choice

Before we describe the second stage of the collective choice procedure, we introduce some notation and definitions. Let $R \subseteq X \times X$ be a binary relation over X. The asymmetric factor $P(R)$ of R and the symmetric factor $I(R)$ of R are defined in the usual way. Given a binary relation R over X and a set $A \in \mathscr{X}$,

[5] The use of the term "nominee" might give the impression that the set of alternatives is equivalent to the set of individuals, but we do not always deal with such a case.

we define the set $G(A, R)$ of all the *greatest elements* in A according to R by

$$G(A, R) = \{x \in A | (x, y) \in R \text{ for all } y \in A\}.$$

For the empty set, we define $G(\emptyset, R) = \emptyset$. Since R is not always reflexive or complete, $G(A, R)$ may not be equivalent to the set $M(A, R)$ of *maximal elements* in A according to R, which is defined as $M(A, R) = \{x \in A | (y, x) \notin P(R) \text{ for all } y \in A\}$.

The *transitive closure* $tc(R)$ of R is defined as follows. For all $x, y \in X$, $(x, y) \in tc(R)$ if and only if there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$ and $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$ and $x^K = y$. It is well known that for any binary relation R, $tc(R)$ is the smallest transitive superset of R.

In the second stage of collective choice, the preferences of voters are aggregated to choose some alternatives from those nominated in the first stage of collective choice. Each voter $j \in \mathcal{V}$ expresses a weak preference over X—that is, a reflexive, complete, and transitive binary relation on X. Let R_j be a weak preference for voter j. Let $\mathscr{R}$ be the set of all weak preferences. As discussed in Iwata (2016), the set of admissible preferences for a given voter, which is called a *preference space*, naturally varies according to the opinion profiles expressed in advance. Therefore, given an opinion profile $J \in \mathscr{J}$, let $\mathcal{D}_j^J \subseteq \mathscr{R}$ be voter j's preference space under opinion profile J. Since a voter who is not a nominator does not express his or her opinion, it is reasonable that his or her preference space is *unrestricted* for all opinion profiles—that is, $\mathcal{D}_j^J = \mathscr{R}$ for all $J \in \mathscr{J}$ and all $j \in \mathcal{V} \backslash \mathcal{M}$.

By contrast, a nominating voter's preference space is restricted because it is natural to consider that his or her preference is correlated with his or her opinion. That is, his or her preference space should be consistent with his or her opinion expressed in the first stage of collective choice. More precisely, it is assumed that every nominating voter always prefers positive alternatives to neutral ones and neutral alternatives to negative ones according to his or her opinion.

Formally, for all $j \in \mathcal{M} \cap \mathcal{V}$, nominating voter j's preference space $\mathcal{D}_j^J$ is restricted in the following way. Given an opinion profile $J \in \mathscr{J}$, let us define a partition $(X_j^+, X_j^0, X_j^-)^J$ of X in such a way that for all $x_k \in X$, $J_{jk} = 1$ implies that x_k is in X_j^+; $J_{jk} = 0$ implies that x_k is in X_j^0; and $J_{jk} = -1$ implies that x_k is in X_j^-. That is, X_j^+ is the set of positive alternatives for j, X_j^0 is the set of neutral alternatives for j, and X_j^- is the set of negative alternatives for j. Then, nominating voter j's preference space $\mathcal{D}_j^J$ is defined as follows:

$$\mathcal{D}_j^J = \{R_j \in \mathscr{R} | \text{for all } x \in X_j^+, \text{ all } y \in X_j^0, \text{ and all } z \in X_j^-, (x, y), (y, z) \in P(R_j)\}.$$

Because R_j is transitive, we have $(x, z) \in P(R_j)$ from the definition of $\mathcal{D}_j^J$ for all $j \in \mathcal{M} \cap \mathcal{V}$.

Given the non-empty subset A of X and opinion profile $J \in \mathscr{J}$, let $T_j^J(A) = \{x \in A | \exists R_j \in \mathcal{D}_j^J \text{ such that } x \in G(A, R_j)\}$. That is, $T_j^J(A)$ is the set of all alternatives x in A such that x is one of the greatest elements according to some voter j's preference $R_j \in \mathcal{D}_j^J$. Iwata (2016) calls $T_j^J(A)$ voter j's *top set of alternatives* in A under J. If a voter j is not a nominator, we have $T_j^J(A) = A$ for all $A \in \mathscr{X}$ because his or her preference space $\mathcal{D}_j^J$ is not restricted at all.

A *preference domain* is a Cartesian product of voters' preference spaces $\mathcal{D}^J = \prod_{j \in \mathcal{V}} \mathcal{D}_j^J$. Let a *preference profile* $\mathbf{R} = (R_1, R_2, \ldots, R_v)$ be v-tuple voters' preferences. Given a nomination rule f and opinion profile $J \in \mathscr{J}$, Iwata (2016) describes the second stage of collective choice as a *collective choice correspondence* (CCC) *with* $f(J)$, $C_f : \mathcal{D}^J \twoheadrightarrow f(J)$. A CCC with $f(J)$ is a mapping as follows. If $f(J) \neq \emptyset$, it assigns a non-empty subset of the nominated alternatives $f(J)$ to each preference profile $\mathbf{R} \in \mathcal{D}^J$; if $f(J) = \emptyset$, it assigns the empty set. Let $C_f(\mathbf{R}, f(J))$ denote the *choice set* of $f(J)$ under $\mathbf{R} \in \mathcal{D}^J$.

Finally, both the first stage and the second stage of collective choice are defined as a *collective choice rule* (CCR) *with* f—namely, a family of CCCs with $f(J)$, $\left\{C_f : \mathcal{D}^J \twoheadrightarrow f(J)\right\}_{J \in \mathscr{J}}$.

3 A Difficulty in Rationalizable Collective Choice Rules

In this section, we point out that any *rationalizable* CCR with f has the difficulty explained in the Introduction. We see that even the minimal condition of collective rationality is too demanding in the two-stage collective choice procedures described in Section 2.

Let $F : \cup_{J \in \mathscr{J}} \mathcal{D}^J \to X \times X$ be a *collective decision function.* A collective decision function F associates a binary relation over X, which is interpreted as a social ranking on X, with each possible preference profile in the union of preference domains for all opinion profiles. If all such social rankings are orderings, which are complete and transitive binary relations, on all alternatives, then it is possible to interpret F as an Arrovian social welfare function.

To incorporate a collective decision function F with the minimal condition of collective rationality, we rely upon results from revealed preference theory. Although revealed preference theory has developed in the environments of individual choice, the results can be easily translated into collective choice settings.[6] In particular, we consider the *general domain* case, where any collection of the subsets of $\mathscr{X} \cup \{\emptyset\}$ can be the possible family of the sets of nominated alternatives. This is because we do not restrict the nomination rules, and therefore any collection of the sets of nominated alternatives is potentially attainable.

We now propose the minimal requirement of collective rationality. A CCR with f is *rationalized with respect to the greatest elements*—*G-rationalized* for short—by a collective decision function F if for all $J \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^J$, $C_f(\mathbf{R}, f(J)) = G(f(J), F(\mathbf{R}))$. That is, for all opinion profiles $J \in \mathscr{J}$ and all preference profiles $\mathbf{R}$ in the corresponding preference domain $\mathcal{D}^J$, the choice set is identified with the set of all the greatest elements in $f(J)$ according to the social ranking $F(\mathbf{R})$. If such a collective decision function F exists, then we may

[6]For example, see Bossert and Suzumura (2010), Deb (2010), and Suzumura (1983).

say that C_f is *G-rationalizable*. G-rationalizability is based on the concept that the finally chosen alternatives weakly dominate all the nominated alternatives according to the social ranking. It is possible to define various notions of rationalizability by imposing additional properties such as completeness or acyclicity on the social ranking $F(\mathbf{R})$. G-rationalizability can be seen as the minimal requirement of collective rationality in two-stage collective choice procedures. In fact, it is possible to show that G-rationalizability is the mildest concept in any notion of rationalizability by adopting similar arguments to the results in terms of the relationships between the various notions of rationalizability proposed in revealed preference theory.[7]

We now point out a difficulty in G-rationalizable CCRs with f. We consider the following property referred to in Iwata (2016). The choice set must remain the same if the set of nominated alternatives (nominees) remains the same when an opinion profile changes. Formally, we define the property as "nominee invariance:"[8]

"*Nominee invariance.*" For all $J^1, J^2 \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^{J^1} \cap \mathcal{D}^{J^2}$, if $f(J^1) = f(J^2)$, then $C_f(\mathbf{R}, f(J^1)) = C_f(\mathbf{R}, f(J^2))$.

This property is trivially satisfied in the standard Arrovian collective choice model because, by definition, if the feasible set is identical, then the choice set is also identical. However, whether nominee invariance is satisfied is not trivial in two-stage collective choice procedures.

We argue that nominee invariance is too demanding in two-stage collective choice procedures because this property means that the information about the opinions of nominators cannot be used in the second stage of collective choice after determining the set of nominees in the first stage of collective choice. On the contrary, it is natural to consider that the information about the opinions of nominators is used in the second stage of collective choice as well as in the first stage of collective choice. As mentioned in the Introduction, the information in recommendation letters and referees' reports is used not only to determine the list of nominees (e.g., as a nomination qualification), but also to make a collective choice from the list of nominees (e.g., as a reference to the final choice). That is, any information about the recommenders and referees can play an important role in making a collective decision in both the first stage and the second stage of collective choice. Nevertheless, nominee invariance prevents any information about the opinions of nominators from being reused in the second stage of collective choice after the set of nominees is determined. This is why we argue that nominee invariance is too demanding in two-stage collective choice procedures.

Not surprisingly, we can show that the following result holds: if a CCR with f violates nominee invariance, then it is not G-rationalizable. That is, nominee

[7]See Bossert and Suzumura (2009, 2010) for the various notions of rationalizability in revealed preference theory and comprehensive logical relationships among them.

[8]Because we do not regard nominee invariance as a reasonable property to impose on CCRs with f, we introduce this property in quotation marks.

invariance is a necessary condition for a CCR with f to be G-rationalizable. To understand this result, suppose that C_f violates nominee invariance. Then, there exist $J^1, J^2 \in \mathscr{J}$ and $\mathbf{R} \in \mathcal{D}^{J^1} \cap \mathcal{D}^{J^2}$ such that $f(J^1) = f(J^2)$ but $C_f(\mathbf{R}, f(J^1)) \neq C_f(\mathbf{R}, f(J^2))$. If C_f is G-rationalized by a collective decision function F, then we have $C_f(\mathbf{R}, f(J^1)) = G(f(J^1), F(\mathbf{R})) = G(f(J^2), F(\mathbf{R})) = C_f(\mathbf{R}, f(J^2))$, which is a contradiction. Thus, G-rationalizability implies nominee invariance. We summarize these observations in Theorem 1.

Theorem 1. *If a CCR with f is G-rationalized by a collective decision function F, then it satisfies nominee invariance.*

Theorem 1 shows that even the minimal condition of collective rationality—namely, G-rationalizability—is too demanding in two-stage collective choice procedures because it implies nominee invariance. Thus, Theorem 1 motivates us to investigate a more moderate concept of rationalizability than G-rationalizability. One possible approach to this issue is to consider the concept of conditional or bounded rationalizability in two-stage collective choice procedures. In comparison with models of choice with bounded rationality, it may be possible to regard the violation of nominee invariance as collective choice being *framed* by the opinions of nominators.[9] That is, according to the manner of expressing the opinions of nominators, the choice set can vary even when the set of nominated alternatives and the preferences of voters do not change at all.

The problem is that G-rationalizability requires that according to the social ranking, the finally chosen alternatives dominate *all* alternatives in the set of nominated alternatives. As pointed out by Bossert and Suzumura (2010), this requirement is difficult to satisfy in some environments, especially in collective choice problems. In two-stage collective choice procedures, this requirement causes the difficulty of using any information about the opinions of nominators in the second stage of collective choice. This is because some nominated alternatives may be qualitatively different from other nominated alternatives according to the opinions of nominators; nevertheless, the difference cannot be reflected by the requirement of G-rationalizability.

In the next section, we consider that the opinions of nominators influence collective decision-making in the second stage of collective choice and propose a moderate concept of rationalizability—conditional or bounded rationalizability—in two-stage collective choice procedures.[10]

4 Nominee-Efficient Collective Choice

In this section, we introduce the conditional (on the opinions of nominators) concept of the rationalizability of CCRs with f. As discussed in Section 3,

[9]See Salant and Rubinstein (2008) for a model of choice with framing effects.

[10]See Bossert and Sprumont (2003) for a concept of rationalizability in terms of the profiles of individuals' preferences rather than the social rankings, within an extended collective choice framework where the alternatives chosen from the set of feasible alternatives may depend on a reference alternative.

the problem is that G-rationalizability requires that all alternatives in the set of nominated alternatives should be dominated by the finally chosen alternatives according to the social ranking. Instead, we investigate *nominee-efficient collective choice* and propose a modest concept of rationalizability.

Let us consider that the society focuses on nominated alternatives that are efficient in terms of the opinions of nominators, instead of all nominated alternatives. An alternative x is *nominee-efficient* according to an opinion profile if it is nominated and there exists no other nominated alternative y such that every nominator i evaluates y as high as x and at least one nominator i' evaluates y higher than x. It is clear that the concept of nominee efficiency is similar to Pareto efficiency according to the preferences of voters. Let $f_E(J) \subseteq f(J)$ denote the set of nominated alternatives that are nominee-efficient under an opinion profile $J \in \mathscr{J}$ and be formally defined as follows. For all $J \in \mathscr{J}$,

$$\begin{aligned} f_E(J) = \{x_k \in f(J) | \nexists x_l \in f(J) \text{ such that } J_{il} \geq J_{ik} \text{ for all } i \in \mathcal{M} \\ \text{and } J_{i'l} > J_{i'k} \text{ for some } i' \in \mathcal{M}\}. \end{aligned}$$

Then, we define that a CCR with f is *nominee-efficient* if and only if we have $C_f(\mathbf{R}, f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^J$. That is, only nominee-efficient alternatives can be collectively chosen from the set of nominated alternatives for all opinion profiles and all preference profiles.

Next, we propose the conditional (on opinion profiles) concept of the rationalizability of CCRs with f in two-stage collective choice procedures. We define that a (nominee-efficient) CCR with f is *nominee-efficiently rationalized—NE-rationalized* for short—by a collective decision function F if for all $J \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^J$,

$$C_f(\mathbf{R}, f(J)) = G(f_E(J), F(\mathbf{R})).$$

We say that C_f is *NE-rationalizable* if such a collective decision function F exists. NE-rationalizability requires that any choice set consists of all the greatest elements in the set of nominee-efficient alternatives according to the social ranking. Therefore, an alternative that is finally chosen, by a CCR with f, from the set of nominated alternatives does not need to be compared with all nominated alternatives, but only with all nominee-efficient alternatives, by the social ranking. One possible interpretation of NE-rationalizability is that any inefficient nominees are already removed from the list of nominees before voters express their preferences, and therefore the finally chosen alternatives do not need to dominate the removed ones according to the social ranking associated with the preferences of voters. Thus, the final choice must be made from the set of nominee-efficient alternatives under the given opinion profile. In this study, we consider a stronger version of NE-rationalizability. A (nominee-efficient) CCR with f is NE-rationalized by a *transitive-valued* collective decision function F if it is NE-rationalized by F such that $F(\mathbf{R})$ is transitive for all $\mathbf{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$.

We now provide an axiomatic foundation of the above two versions of conditional rationalizability. Let us define the *nominee-efficient direct revealed preference relation* as follows:

$$R^E_{C_f^{\mathbf{R}}} = \{(x, y) \in X \times X | \exists J \in \mathscr{J} \text{ such that } \mathbf{R} \in \mathcal{D}^J,\ x \in C_f(\mathbf{R}, f(J)), \text{ and } y \in f_E(J)\}.$$

The basic idea behind the nominee-efficient direct revealed preference relation is similar to the direct revealed preference relation discussed in the revealed preference literature. That is, if x is collectively chosen from the set of nominated alternatives when y is nominee-efficient under the opinion profile, then x is revealed to be at least as good as y.

The next result is a counterpart of the result presented by Samuelson (1938, 1948) in revealed preference theory and shows that if a CCR with f is NE-rationalized by a collective decision function F, then any social ranking $F(\mathbf{R})$ must respect the nominee-efficient direct revealed preference relation.

Lemma 1. *If a CCR with f is NE-rationalized by a collective decision function F, then $R^E_{C_f^{\mathbf{R}}} \subseteq F(\boldsymbol{R})$ for all $\boldsymbol{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$.*

Proof. Suppose that a CCR with f is NE-rationalized by a collective decision function F and $x, y \in X$ are such that $(x, y) \in R^E_{C_f^{\mathbf{R}}}$ for any $\mathbf{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$. From the definition of $R^E_{C_f^{\mathbf{R}}}$, there exists $J \in \mathscr{J}$ such that $\mathbf{R} \in \mathcal{D}^J$, $x \in C_f(\mathbf{R}, f(J))$, and $y \in f_E(J)$. Because C_f is NE-rationalized by F, we obtain $(x, y) \in F(\mathbf{R})$. Therefore, we have $R^E_{C_f^{\mathbf{R}}} \subseteq F(\mathbf{R})$. $\square$

The next result is an extension of Lemma 1 to the case where a CCR with f is NE-rationalized by a transitive-valued collective decision function. It is shown that if a CCR with f is NE-rationalized by a transitive-valued collective decision function F, then any social ordering $F(\mathbf{R})$ must respect the transitive-closure of the nominee-efficient direct revealed preference relation.

Lemma 2. *If a CCR with f is NE-rationalized by a transitive-valued collective decision function F, then $tc(R^E_{C_f^{\mathbf{R}}}) \subseteq F(\boldsymbol{R})$ for all $\boldsymbol{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$.*

Proof. Suppose that a CCR with f is NE-rationalized by a transitive-valued collective decision function F and $x, y \in X$ are such that $(x, y) \in tc(R^E_{C_f^{\mathbf{R}}})$. From the definition of the transitive closure of $R^E_{C_f^{\mathbf{R}}}$, there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R^E_{C_f^{\mathbf{R}}}$ for all $k \in \{1, \ldots, K\}$, and $x^K = y$. From Lemma 1, we obtain $(x^{k-1}, x^k) \in F(\mathbf{R})$ for all $k \in \{1, \ldots, K\}$. Since $x = x^0$ and $x^K = y$, we have $(x, y) \in F(\mathbf{R})$ owing to the repeated use of the transitivity of $F(\mathbf{R})$. Therefore, we have $tc(R^E_{C_f^{\mathbf{R}}}) \subseteq F(\mathbf{R})$. $\square$

We are now ready to characterize the two properties of the NE-rationalizability of CCRs with f. We combine the concept of direct-revelation coherence, or V-axiom in Richter's (1971) terminology, in revealed preference theory with the notion of nominee efficiency to provide a necessary and sufficient condition for the NE-rationalizability of CCRs with f.

Nominee-efficient direct-revelation coherence. For all $J \in \mathscr{J}$, all $x \in f_E(J)$, and all $\mathbf{R} \in \mathcal{D}^J$,

$$(x, y) \in R^E_{C_f^{\mathbf{R}}} \text{ for all } y \in f_E(J) \Rightarrow x \in C_f(\mathbf{R}, f(J)).$$

Nominee-efficient direct-revelation coherence requires that every nominee-efficient alternative that weakly dominates any other nominee-efficient alternative according to the nominee-efficient direct revealed preference relation must be collectively chosen. In fact, Theorem 2 shows that this property is a necessary and sufficient condition for a CCR with f to be NE-rationalized by a collective decision function F.

Theorem 2. *A CCR with f is NE-rationalized by a collective decision function F if and only if it satisfies nominee-efficient direct-revelation coherence.*

Proof. We first prove the only-if part of Theorem 2. Suppose that a CCR with f is NE-rationalized by a collective decision function F. Let $J \in \mathscr{J}$, $x \in f_E(J)$, and $\mathbf{R} \in \mathcal{D}^J$ be such that $(x, y) \in R^E_{C_f^{\mathbf{R}}}$ for all $y \in f_E(J)$. From Lemma 1, $(x, y) \in F(\mathbf{R})$ holds for all $y \in f_E(J)$, which implies $x \in C_f(\mathbf{R}, f(J))$ because C_f is NE-rationalized by F.

Next, we turn to the if part of Theorem 2. Suppose that C_f satisfies nominee-efficient direct-revelation coherence. We show that C_f is NE-rationalized by a collective decision function F by means of setting $F(\mathbf{R}) = R^E_{C_f^{\mathbf{R}}}$ for all $\mathbf{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$. Let $J \in \mathscr{J}$, $x \in f_E(J)$, and $\mathbf{R} \in \mathcal{D}^J$. Suppose $x \in C_f(\mathbf{R}, f(J))$. From the definition of $R^E_{C_f^{\mathbf{R}}}$, we have $(x, y) \in R^E_{C_f^{\mathbf{R}}} = F(\mathbf{R})$ for all $y \in f_E(J)$, which implies that $x \in G(f_E(J), F(\mathbf{R}))$.

Conversely, suppose that $(x, y) \in R^E_{C_f^{\mathbf{R}}} = F(\mathbf{R})$ for all $y \in f_E(J)$, which implies that $x \in G(f_E(J), F(\mathbf{R}))$. It follows from nominee-efficient direct-revelation coherence that $x \in C_f(\mathbf{R}, f(J))$. Thus, we have shown that C_f is NE-rationalized by F. $\square$

By using an argument in the revealed preference literature, it is possible to show that NE-rationalizability is equivalent to the NE-rationalizability of CCRs with f by a *reflexive-valued* collective decision function F, under which any social ranking is reflexive over X. For readers interested in this result and its proof, see Richter (1971).

Next, we characterize the NE-rationalizability of CCRs with f by a transitive-valued collective decision function F. The required property is obtained by replacing $R^E_{C_f^{\mathbf{R}}}$ in the definition of nominee-efficient direct-revelation coherence with its transitive closure $tc(R^E_{C_f^{\mathbf{R}}})$.

Nominee-efficient transitive-closure coherence. For all $J \in \mathscr{J}$ and all $x \in f_E(J)$, and all $\mathbf{R} \in \mathcal{D}^J$,

$$(x, y) \in tc(R^E_{C_f^{\mathbf{R}}}) \text{ for all } y \in f_E(J) \Rightarrow x \in C_f(\mathbf{R}, f(J)).$$

Then, we obtain the following result.

Theorem 3. *A CCR with f is NE-rationalized by a transitive-valued collective decision function F if and only if it satisfies nominee-efficient transitive-closure coherence.*

Proof. We can prove Theorem 2 by a similar argument to the proof of Theorem 1. What is needed is to replace $R^E_{C^{\mathbf{R}}_f}$ with $tc(R^E_{C^{\mathbf{R}}_f})$ and use Lemma 2 instead of Lemma 1. $\square$

By using a similar argument to the result of Richter (1966, 1971) in the revealed preference literature, it is possible to show that the NE-rationalizability of CCRs with f by a transitive-valued collective decision function F is equivalent to the NE-rationalizability of CCRs with f by a collective decision function F, under which any social ranking is an ordering over X. We omit the proof of this result.

Finally, we conclude this section with a few comments. In addition to the above two properties of NE-rationalizability, we can consider further alternative concepts of the NE-rationalizability of CCRs with f. For example, we can say that a CCR with f is NE-rationalized by a Suzumura-consistent-valued (Bossert and Suzumura 2008 and Suzumura 1976, 1983) collective decision function F if it is NE-rationalized by F such that $F(\mathbf{R})$ is Suzumura-consistent for all $\mathbf{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$. Then, it is possible to characterize the NE-rationalizability of CCRs with f by a Suzumura-consistent-valued collective decision function F in a similar way to the proofs of Theorems 2 and 3.

5 The Possibility of Nominee-Efficient Collective Choice: A Characterization

In this section, we provide a characterization of the nomination rules under which a CCR with f satisfies Arrow's well-known axioms as well as nominee-efficient transitive-closure coherence discussed in Section 4. Arrow's axioms in two-stage collective choice procedures are found in Iwata (2016), and we omit further explanations here.

Weak Pareto. For all $J \in \mathscr{J}$, all $x, y \in f(J)$, and all $\mathbf{R} \in \mathcal{D}^J$, if for all $j \in \mathcal{V}$, $(x, y) \in P(R_j)$, then $y \notin C_f(\mathbf{R}, f(J))$.

Independence. For all $J \in \mathscr{J}$, all $x, y \in f(J)$, and all $\mathbf{R}^1, \mathbf{R}^2 \in \mathcal{D}^J$, if $\mathbf{R}^1|_{\{x,y\}} = \mathbf{R}^2|_{\{x,y\}}$, $x \in C_f(\mathbf{R}^1, f(J))$, and $y \notin C_f(\mathbf{R}^1, f(J))$, then $y \notin C_f(\mathbf{R}^2, f(J))$.

Given an opinion profile $J \in \mathscr{J}$, a voter d is a *dictator on* $f(J)$ if for all $x, y \in f(J)$ and all $\mathbf{R} \in \mathcal{D}^J$, $(x, y) \in P(R_d)$ implies that $y \notin C_f(\mathbf{R}, f(J))$. A CCC with $f(J)$ is *dictatorial* if there exists a dictator on $f(J)$. A CCR with f is dictatorial if there exists a dictator on $f(J)$ for all $J \in \mathscr{J}$.

Non-dictatorship. There exists no dictator on $f(J)$ for all $J \in \mathscr{J}$.

Before we state our main theorem, some definitions are provided. For all $J, J' \in \mathscr{J}$, J is *directly related to* J' if there exist $K \in \mathbb{N}$, $x^0, \dots, x^K \in X$, and $J^0, \dots, J^K \in \mathscr{J}$ such that the following four conditions hold:

1. $J' = J^0$, and either $J = J^1$ or $J = J^K$,

2. For all $j \in \mathcal{V}$, $x, y \in T_j^{J^0}(f(J^0))$ with $x = x^0$ and $x^K = y$,

3. For all $j \in \mathcal{V}$, $x^{k-1}, x^k \in T_j^{J^k}(f(J^k))$ for all $k \in \{1, \ldots, K\}$,

4. For all $\mathbf{R} \in \cap_{k \in \{0,\ldots,K\}} \mathcal{D}^{J^k}$, there exists $\mathbf{R}' \in \cap_{k \in \{0,\ldots,K\}} \mathcal{D}^{J^k}$ such that $\mathbf{R}|_{\{x,y\}} = \mathbf{R}'|_{\{x,y\}}$ and for all $j \in \mathcal{V}$, either of the two conditions holds;

   (a) If $(x, y) \in P(R_j)$, then $x^{k-1} \in G(T_j^{J^k}(f(J^k)), R'_j)$ for all $k \in \{1, \ldots K\}$, or

   (b) If $(y, x) \in R_j$, then $x^{k-1} \in G(T_j^{J^k}(f(J^k)), R'_j)$ for all $k \in \{2, \ldots K\}$.

For all $J, J' \in \mathscr{J}$, J is *indirectly related to* J' if there exist $K \in \mathbb{N}$ and $J^0, \ldots, J^K \in \mathscr{J}$ such that $J = J^0$, $J^K = J'$, and J^{k-1} is directly related to J^k for all $k \in \{1, \ldots, K\}$. Given an opinion profile $J \in \mathscr{J}$, let $\mathscr{J}^J \subseteq \mathscr{J}$ be the set of opinion profiles that consists of J and all $J' \in \mathscr{J}$ such that J is indirectly related to J'.

For all $J, J' \in \mathscr{J}$, J is *connected with* J' if there exist $K \in \mathbb{N}$ and $\mathscr{J}^{J^0}, \ldots, \mathscr{J}^{J^K} \subseteq \mathscr{J}$ such that $J \in \mathscr{J}^{J^0}$, $J' \in \mathscr{J}^{J^K}$, and $\mathscr{J}^{J^{k-1}} \cap \mathscr{J}^{J^k} \neq \emptyset$ for all $k \in \{1, \ldots, K\}$. For all $J, J' \in \mathscr{J}$, J is *strongly connected with* J' if there exist $K \in \mathbb{N}$ and $\mathscr{J}^{J^0}, \ldots, \mathscr{J}^{J^K} \subseteq \mathscr{J}$ with $\#T_j^{J^k}(f(J^k)) \geq 3$ for all $j \in \mathcal{V}$ and all $k \in \{1, \ldots, K\}$ such that $J \in \mathscr{J}^{J^0}$, $J' \in \mathscr{J}^{J^K}$, and $\mathscr{J}^{J^{k-1}} \cap \mathscr{J}^{J^k} \neq \emptyset$ for all $k \in \{1, \ldots, K\}$.

We are now ready to provide our main contribution that identifies a necessary and sufficient condition for the nomination rules under which a CCR with f satisfies Arrow's axioms and nominee-efficient transitive-closure coherence.

Theorem 4. *There exists a CCR with f satisfying weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence if and only if f satisfies both of the following conditions:*

1. *$\cap_{j \in \mathcal{V}} T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$.*

2. *Either of the following four conditions holds;*

   (a) $T_j^J(f(J)) \nsubseteq f_E(J)$ for some $J \in \mathscr{J}$ and some $j \in \mathcal{V}$,

   (b) $T_i^J(f(J)) \neq T_j^J(f(J))$ for some $J \in \mathscr{J}$ and some $i, j \in \mathcal{V}$,

   (c) $\#T_j^J(f(J)) < 3$ for all $J \in \mathscr{J}$ and all $j \in \mathcal{V}$, and for some $J' \in \mathscr{J}$, $\#T_j^{J'}(f(J')) = 2$ for all $j \in \mathcal{V}$,

   (d) There exists $J \in \mathscr{J}$ with $\#T_j^J(f(J)) \geq 3$ for all $j \in \mathcal{V}$ such that J is not strongly connected with some $J' \in \mathscr{J}$ with $\#T_j^J(f(J)) \geq 2$ for all $j \in \mathcal{V}$.

We make a few remarks on the two conditions found in Theorem 4. First, we do not argue that conditions 1 and 2 in Theorem 4 are reasonable from a normative viewpoint; rather these conditions work as a checklist for the nomination rules to verify whether nominee-efficient collective choice is compatible with Arrow's axioms under a nomination rule. Condition 1 in Theorem 4 requires that alternatives in the top set of alternatives for all voters must be nominee-efficient. If this condition is violated, then there exists a preference profile such that every nominee-efficient alternative is weakly Pareto-dominated by a nominee-inefficient alternative. Therefore, nominee-efficient collective choice is incompatible with weak Pareto. In addition, the essence of the four conditions of condition 2 in Theorem 4 is as follows:

(a) At least one voter exists whose top set of alternatives is not included in the set of nominee-efficient alternatives,

(b) A diversity of the top set of alternatives exists among at least two voters,

(c) Every voter's top set of alternatives degenerates in the sense that it contains at most two alternatives,

(d) There exists at least one opinion profile that is not strongly "connected" with another opinion profile.

We now illustrate how those conditions for the nomination rules in Theorem 4 work by considering specific and reasonable nomination rules. First, we define a nomination rule f as the *unanimous rule* if for all $J \in \mathscr{J}$ and all $x_k \in X$, $x_k \in f(J)$ if and only if $J_{ik} = 1$ for all $i \in \mathcal{M}$. We investigate whether the unanimous rule satisfies conditions 1 and 2 in Theorem 4. Since we have $f(J) = f_E(J)$ for all $J \in \mathscr{J}$, it is clear that the unanimous rule satisfies condition 1. In addition, it is easy to check that conditions 2 (a) and 2 (b) fail for the unanimous rule. Because we have $f(J) = X$ for the opinion profile J such that $J_{ik} = 1$ for all $i \in \mathcal{M}$ and all $x_k \in X$ and $\mathcal{D}^J$ is an unrestricted preference domain, the unanimous rule violates both conditions 2 (c) and 2 (d). Therefore, we obtain the following result.

Corollary 1. *Suppose that f is the unanimous rule. Then, there exists no CCR with f satisfying weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence.*[11]

On the contrary, when we consider a stronger version of the unanimous rule, the result is different from that of the unanimous rule. Define a nomination rule f as the *strong unanimous rule* if for all $J \in \mathscr{J}$ and all $x_k \in X$, $x_k \in f(J)$ if and only if $J_{ik} \geq 0$ for all $i \in \mathcal{M}$. First, suppose that nominator $i \in \mathcal{M}$ is not a voter. Then, the strong unanimous rule f violates condition 1 in Theorem 4. Consider an opinion profile $J \in \mathscr{J}$ such that $J_{i1} = 1$ and $J_{ik} = 0$ for all $x_k \in X \backslash \{x_1\}$, and $J_{jk} = 0$ for all $j \in \mathcal{M} \backslash \{i\}$ and all $x_k \in X$. Then, from

[11] In fact, Corollary 1 is valid for any subrule of the unanimous rule such that $f(J) = X$ for the opinion profile $J \in \mathscr{J}$ with $J_{ik} = 1$ for all $i \in \mathcal{M}$ and all $x_k \in X$.

the construction of f, we have $f(J) = X$ and $f_E(J) = \{x_1\}$. Since $\mathcal{D}^J$ is unrestricted, it is clear that condition 1 does not hold. In fact, suppose that C_f satisfies weak Pareto, and consider a preference profile $\mathbf{R} \in \mathcal{D}^J$ such that $(x_2, x_1) \in P(R_j)$ for all $j \in \mathcal{V}$. By weak Pareto, we have $x_1 \notin C_f(\mathbf{R}, f(J))$, which implies that C_f is not NE-rationalized by a transitive-valued collective decision function F, and therefore it violates nominee-efficient transitive-closure coherence from Theorem 3.

Thus, the existence of a nominator who does not vote is not compatible with condition 1 in Theorem 4 under the strong unanimous rule. However, if every nominator is a voter, then it is easily possible to check that the strong unanimous rule satisfies condition 1 in Theorem 4. Moreover, if a voter is not a nominator, then condition 2 (a) is also satisfied. In addition, if every individual is a nominating voter, then the strong unanimous rule violates condition 2 (a) but satisfies condition 2 (b). Therefore, from Theorem 4, we can find a CCR with f satisfying the four axioms in Theorem 4. Thus, we obtain the following result.

Corollary 2. *Suppose that f is the strong unanimous rule. Then, there exists a CCR with f satisfying weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence if and only if $\mathcal{M} \subseteq \mathcal{V}$.*

6 Concluding Remarks

This study examined the validity of the collective rationality condition in two-stage collective choice procedures. We first showed that even the minimal collective rationality condition (G-rationalizability) is too demanding for two-stage collective choice procedures because it implies nominee invariance, which requires that the opinions of nominators are irrelevant to the second stage of collective choice (Theorem 1).

We alternatively proposed nominee-efficient collective choice in which only nominee-efficient alternatives are collectively chosen from the set of nominated alternatives. Thus, nominee-efficient collective choice depends on the opinions of nominators in the second stage of collective choice, which attracts our attention to the exploration of the conditional rationality condition in collective choice. We proposed a new concept of conditional (on the opinions of nominators) rationalizability, called NE-rationalizability, and provided a necessary and sufficient condition for two-stage collective choice procedures to be NE-rationalizable (Theorems 2 and 3). Our main contribution is to characterize the class of the nomination rules under which nominee-efficient collective choice is compatible with Arrow's axioms (Theorem 4). Theorem 4 is particularly important because it identifies a crucial boundary between the possibility and impossibility of two-stage collective choice procedures.

We conclude this paper with final remarks. In this study, we did not discuss the desirability of the nomination rules from a normative viewpoint, and a nomination rule was exogenously fixed in two-stage collective choice procedures. A question posed here is how to evaluate the nomination rules from a normative viewpoint. When a nomination rule is dictatorial, the resulting state is undesirable even if it satisfies conditions 1 and 2 in Theorem 4. One possible approach to this issue is to evaluate the nomination rules based on the voting power distributions that characterize them. In fact, the nomination rules are technically relevant to ternary voting games (Felsenthal and Machover 1997) with multiple issues when we interpret a neutral opinion in this study as "abstention" discussed in the voting game literature.[12] In a recent study, Iwata (2018) proposes a two-stage aggregation procedure to rank the nomination rules based on the distributions of voting powers, where dictatorial rules are associated with the lowest ranking of the nomination rules.

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Appendix

In this appendix, we provide the proof of Theorem 4. The following lemmas are useful for proving Theorem 4.

Lemma 3. *Suppose that there exist $J \in \mathscr{J}$ and $x \in f(J)$ such that $x \in T_j^J(f(J))$ for all $j \in \mathcal{V}$ and $x \notin f_E(J)$. Then, there exists no CCR with f satisfying weak Pareto and nominee-efficient transitive-closure coherence.*

Proof. Consider a nomination rule f such that there exist $J \in \mathscr{J}$ and $x \in f(J)$ with $x \in T_j^J(f(J))$ for all $j \in \mathcal{V}$ and $x \notin f_E(J)$. Suppose that a CCR with f satisfies weak Pareto and nominee-efficient transitive-closure coherence. Then, there exists $\mathbf{R} \in \mathcal{D}^J$ such that $\{x\} = G(f(J), R_j)$ for all $j \in \mathcal{V}$. By weak Pareto, we have $\{x\} = C_f(\mathbf{R}, f(J))$. Since the CCR with f satisfies nominee-efficient transitive-closure coherence, it follows from Theorem 3 that it is NE-rationalized by a transitive-valued collective decision function F. Therefore, we have $C_f(\mathbf{R}, f(J)) = G(f_E(J), F(\mathbf{R})) \subseteq f_E(J)$, which is a contradiction because we have $x \notin f_E(J)$. □

Lemma 4. *Suppose that $\cap_{j\in\mathcal{V}} T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$. If there exist $J \in \mathscr{J}$ and $i \in \mathcal{V}$ such that $T_i^J(f(J)) \nsubseteq f_E(J)$, then there exists a CCR with f satisfying weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence.*

[12] See also Ju (2010) for the relationship between the two collective decision-making procedures.

Proof. Construct the following CCR with f. For all $J \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^J$, $C_f(\mathbf{R}, f(J)) = G(f_E(J), R_i)$. It is clear that C_f is well defined and satisfies independence.

On the contrary, suppose that C_f violates weak Pareto. Then, there exist $J \in \mathscr{J}$, $x_k, x_l \in f(J)$, and $\mathbf{R} \in \mathcal{D}^J$ such that $(x_k, x_l) \in P_j$ for all $j \in \mathcal{V}$ and $x_l \in C_f(\mathbf{R}, f(J))$. From $(x_k, x_l) \in P_i$ and the construction of C_f, we have $x_k \notin f_E(J)$ and $x_l \in f_E(J)$. If no nominating voter exists, we have $f(J) = f_E(J)$ by $\cap_{j \in \mathcal{V}} T_j^J(f(J)) \subseteq f_E(J)$, which contradicts $x_k \notin f_E(J)$ and $x_l \in f_E(J)$. If there exists a nominating voter, we have $J_{jk} = J_{jl}$ for all $j \in \mathcal{M} \cap \mathcal{V}$ because we have $x_k \notin f_E(J)$, $x_l \in f_E(J)$, and $(x_k, x_l) \in P_j$ for all $j \in \mathcal{V}$. From $\cap_{j \in \mathcal{V}} T_j^J(f(J)) \subseteq f_E(J)$, we have $x_k \in f_E(J)$ if and only if $x_l \in f_E(J)$, which is a contradiction. Therefore, C_f satisfies weak Pareto.

We now show that C_f satisfies non-dictatorship. Since, by assumption, there exist $J \in \mathscr{J}$ and $x \in f(J)$ such that $x \in T_i^J(f(J))$ and $x \notin f_E(J)$, we have $\mathbf{R} \in \mathcal{D}^J$ such that $\{x\} = G(f(J), R_i)$. From the construction of C_f, we have $x \notin C_f(\mathbf{R}, f(J))$, which implies that i is not a dictator on $f(J)$. It is easy to show that any other voter is not a dictator. Therefore, C_f satisfies non-dictatorship.

Finally, we show that C_f satisfies nominee-efficient transitive-closure coherence. It is clear that C_f is NE-rationalized by a transitive-valued collective decision function F by setting $F(\mathbf{R}) = R_i$ for all $\mathbf{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$. From Theorem 3, this satisfies nominee-efficient transitive-closure coherence. $\square$

Lemma 5. *Suppose that $T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$ and all $j \in \mathcal{V}$. If there exist $J \in \mathscr{J}$ and $i, j \in \mathcal{V}$ such that $T_i^J(f(J)) \neq T_j^J(f(J))$, then there exists a CCR with f satisfying weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence.*

Proof. We distinguish two possible cases: (i) $\#T_i^{J'}(f(J')) < 2$ for all $J' \in \mathscr{J}$ and (ii) $\#T_i^{J'}(f(J')) \geq 2$ for some $J' \in \mathscr{J}$.

Case (i) Let $\{x\} = T_i^J(f(J))$. First, suppose that there exists $J' \in \mathscr{J}$ with $\{x\} \neq T_i^{J'}(f(J'))$. Construct the following CCR with f. For all $J'' \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^{J''}$, if $\{x\} = T_i^{J''}(f(J''))$, then $C_f(\mathbf{R}, f(J'')) = G(f_E(J''), R_i)$; and if $\{x\} \neq T_i^{J''}(f(J''))$, then $C_f(\mathbf{R}, f(J'')) = G(f_E(J''), R_j)$ for a fixed voter $j \in \mathcal{V} \backslash \{i\}$. It is clear that C_f is well defined and satisfies independence and non-dictatorship. Since $T_{i'}^{J''}(f(J'')) \subseteq f_E(J'')$ for all $J'' \in \mathscr{J}$ and all $i' \in \mathcal{V}$, C_f satisfies weak Pareto from the construction of C_f.

Next, we show that C_f satisfies nominee-efficient transitive-closure coherence. For all $J'' \in \mathscr{J}$ and all $y \in X \backslash \{x\}$, we have $y \notin f(J'')$ if there exist $\{x\} = T_i^{J''}(f(J''))$ and $R_i \in \mathcal{D}_i^{J''}$ such that $(y, x) \in R_i$; and also we have $x \notin f(J'')$ if there exist $\{y\} = T_i^{J''}(f(J''))$ and $R_i \in \mathcal{D}_i^{J''}$ such that $(x, y) \in R_i$. Therefore, for all $J'', J''' \in \mathscr{J}$ and all $y \in X \backslash \{x\}$ such that $\{x\} = T_i^{J''}(f(J''))$ and $\{y\} = T_i^{J'''}(f(J'''))$, if we have $x, y \in f_E(J'') \cap f_E(J''')$, then we have $\mathcal{D}^{J''} \cap \mathcal{D}^{J'''} = \emptyset$. Then, since we have $T_i^J(f(J)) \subseteq f_E(J)$ and $T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$, C_f is NE-rationalized by a transitive-valued collective decision function F by setting either $F(\mathbf{R}) = R_i$ or $F(\mathbf{R}) = R_j$ for all $\mathbf{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$.

Therefore, this satisfies nominee-efficient transitive-closure coherence from Theorem 3.

Second, suppose that there exists no $J' \in \mathscr{J}$ with $\{x\} \neq T_i^{J'}(f(J'))$. Construct the following CCR with f. For all $J'' \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^{J''}$, $C_f(\mathbf{R}, f(J'')) = (G(X, R_j) \cap f_E(J'')) \cup \{x\}$ for a fixed voter $j \in \mathcal{V} \backslash \{i\}$. It is clear that C_f is well defined and satisfies independence. Since $T_{i'}^{J''}(f(J'')) \subseteq f_E(J'')$ for all $J'' \in \mathscr{J}$ and all $i' \in \mathcal{V}$, C_f satisfies weak Pareto from the construction of C_f.

We now show that C_f satisfies non-dictatorship. Since we have $T_i(f(J)) \neq T_j^J(f(J))$, there exist $y \in T_j^J(f(J))$ and $\mathbf{R} \in \mathcal{D}^J$ such that $(y, x) \in P_j$. From the construction of C_f, we have $x \in C_f(\mathbf{R}, f(J))$, which implies that j is not a dictator on $f(J)$. In addition, since we have $T_i^J(f(J)) \neq T_j^J(f(J))$, there exist $y \in T_j^J(f(J))$ and $\mathbf{R}' \in \mathcal{D}^J$ such that $(x, y) \in P_i'$ and $(y, x) \in P_j'$. From the construction of C_f, we have $y \in C_f(\mathbf{R}', f(J))$, which implies that i is not a dictator on $f(J)$. We can easily show that no other voter is a dictator on $f(J)$. Therefore, C_f satisfies non-dictatorship.

Finally, we show that C_f satisfies nominee-efficient transitive-closure coherence. Let us consider the following binary relation $Q(\mathbf{R})$ over X with respect to $\mathbf{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$ such that $G(X, Q(\mathbf{R})) = G(\cup_{J'' \in \mathscr{J}} f_E(J''), R_j) \cup \{x\}$. That is, $Q(\mathbf{R})$ is such that x is added to the set of the greatest elements in $\cup_{J'' \in \mathscr{J}} f_E(J'')$ according to R_j. Then, since we have $T_i^J(f(J)) \subseteq f_E(J)$ and $T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$, C_f is NE-rationalized by a transitive-valued collective decision function F by setting $F(\mathbf{R}) = Q(\mathbf{R})$ for all $\mathbf{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$. Therefore, it satisfies nominee-efficient transitive-closure coherence from Theorem 3.

Case (ii). Construct the following CCR with f. For all $J \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^J$, $C_f(\mathbf{R}, f(J)) = G(T_i^J(f(J)), R_j)$. C_f is clearly well defined. We can also show that it satisfies weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence by similar arguments to Case (i). $\square$

Lemma 6. *Suppose that $T_i^J(f(J)) = T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$ and all $i, j \in \mathcal{V}$. Then, if f satisfies condition 2 (c) of Theorem 4, then there exists a CCR with f satisfying weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence.*

Proof. Construct the following CCR with f. For all $J \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^J$, $x \in C_f(\mathbf{R}, f(J))$ if and only if the Borda score of x is as high as that of y for all $y \in f(J) \backslash \{x\}$. Because $T_i^J(f(J)) = T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$ and all $i, j \in \mathcal{V}$ and no $J \in \mathscr{J}$ exists such that $\#T_j^J(f(J)) \geq 3$ for all $j \in \mathcal{V}$ from condition 2 (c) in Theorem 4, any alternatives not in $T_i^J(f(J))$ are never chosen, and therefore it is well defined. From the construction of C_f, it satisfies weak Pareto and non-dictatorship. It also satisfies independence because we have $\#T_j^J(f(J)) < 3$ for all $j \in \mathcal{V}$ from condition 2 (c) in Theorem 4.

We now show that C_f satisfies nominee-efficient transitive-closure coherence. Consider a complete and transitive binary relation $Q(\mathbf{R})$ over X with respect to $\mathbf{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$ such that $x \in G(f(J), Q(\mathbf{R}))$ if and only if x has the highest Borda score in $f(J)$. From the assumptions of Lemma 6, we have $C_f(\mathbf{R}, f(J)) = G(f_E(J), Q(\mathbf{R}))$ for all $J \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^J$, which implies C_f is NE-rationalized by a transitive-valued collective decision function F by setting $F(\mathbf{R}) = Q(\mathbf{R})$ for all $\mathbf{R} \in \cup_{J \in \mathscr{J}} \mathcal{D}^J$. Therefore, it satisfies nominee-efficient transitive closure coherence from Theorem 3. $\square$

Before we proceed to the next lemma, we provide some definitions. For all $J, J' \in \mathscr{J}$ and a fixed $i \in \mathcal{V}$, J_i is *directly related to* J'_i if there exist $K \in \mathbb{N}$, $x^0, \dots, x^K \in X$, and $J^0, \dots, J^K \in \mathscr{J}$ such that the following four conditions hold:

1. $J' = J^0$, and either $J = J^1$ or $J = J^K$,
2. $x, y \in T_i^{J^0}(f(J^0))$ with $x = x^0$ and $x^K = y$,
3. $x^{k-1}, x^k \in T_i^{J^k}(f(J^k))$ for all $k \in \{1, \dots, K\}$,
4. For all $R_i \in \cap_{k \in \{0,\dots,K\}} \mathcal{D}_i^{J^k}$, there exists $R'_i \in \cap_{k \in \{0,\dots,K\}} \mathcal{D}_i^{J^k}$ such that $R_i|_{\{x,y\}} = R'_i|_{\{x,y\}}$ and either of the two conditions holds;
   (a) If $(x, y) \in P(R_i)$, then $x^{k-1} \in G(T_i^{J^k}(f(J^k), R'_i)$ for all $k \in \{1, \dots K\}$, or
   (b) If $(y, x) \in R_i$, then $x^{k-1} \in G(T_i^{J^k}(f(J^k), R'_i)$ for all $k \in \{2, \dots K\}$.

For all $J, J' \in \mathscr{J}$ and a fixed $i \in \mathcal{V}$, J_i is *indirectly related to* J'_i if there exist $K \in \mathbb{N}$ and $J^0, \dots, J^K \in \mathscr{J}$ such that $J = J^0$, $J^K = J'$, and J_i^{k-1} is directly related to J_i^k for all $k \in \{1, \dots, K\}$. Given an opinion profile $J \in \mathscr{J}$ and a voter $i \in \mathcal{V}$, let $\mathscr{J}^{J_i} \subseteq \mathscr{J}$ be the set of opinion profiles that consists of J and all $J' \in \mathscr{J}$ such that J_i is indirectly related to J'_i.

For all $J, J' \in \mathscr{J}$ and a fixed $i \in \mathcal{V}$, J_i is *connected with* J'_i if there exist $K \in \mathbb{N}$ and $\mathscr{J}^{J_i^0}, \dots, \mathscr{J}^{J_i^K} \subseteq \mathscr{J}$ such that $J \in \mathscr{J}^{J_i^0}$, $J' \in \mathscr{J}^{J_i^K}$, and $\mathscr{J}^{J_i^{k-1}} \cap \mathscr{J}^{J_i^k} \neq \emptyset$ for all $k \in \{1, \dots, K\}$. For all $J, J' \in \mathscr{J}$ and a fixed $i \in \mathcal{V}$, J_i is *strongly connected with* J'_i if there exist $K \in \mathbb{N}$ and $\mathscr{J}^{J_i^0}, \dots, \mathscr{J}^{J_i^K} \subseteq \mathscr{J}$ with $\#T_j^{J^k}(f(J^k)) \geq 3$ for all $j \in \mathcal{V}$ and all $k \in \{1, \dots, K\}$ such that $J \in \mathscr{J}^{J_i^0}$, $J' \in \mathscr{J}^{J_i^K}$, and $\mathscr{J}^{J_i^{k-1}} \cap \mathscr{J}^{J_i^k} \neq \emptyset$ for all $k \in \{1, \dots, K\}$.

Lemma 7. *Suppose that $T_i^J(f(J)) = T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$ and all $i, j \in \mathcal{V}$. Then, if f satisfies condition 2 (d) of Theorem 4, then there exists a CCR with f satisfying weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence.*

Proof. From condition 2 (d), there exists $J^* \in \mathscr{J}$ with $\#T_j^{J^*}(f(J^*)) \geq 3$ for all $j \in \mathcal{V}$ such that J^* is not strongly connected with some $J^{**} \in \mathscr{J}$ with $\#T_j^{J^{**}}(f(J^{**})) \geq 2$ for all $j \in \mathcal{V}$. We distinguish two possible cases: (i) there exists $J \in \mathscr{J}$ with $\#T_j^J(f(J)) \geq 2$ for all $j \in \mathcal{V}$ such that $J \notin \mathscr{J}^{J'}$ for all $J' \in \mathscr{J}$ with $\#T_j^{J'}(f(J')) \geq 3$ for all $j \in \mathcal{V}$; or (ii) for all $J \in \mathscr{J}$ with

$\#T_j^J(f(J)) \geq 2$ for all $j \in \mathcal{V}$, there exists $J' \in \mathscr{J}$ with $\#T_j^{J'}(f(J')) \geq 3$ for all $j \in \mathcal{V}$ such that $J \in \mathscr{J}^{J'}$.

Case (i). For all $J \in \mathscr{J} \setminus \bigcup_{\#T_j^{J'}(f(J')) \geq 3} \mathscr{J}^{J'}$, we have $\#T_j^J(f(J)) \leq 2$ for all $j \in \mathcal{V}$ by definition.

Let $\Xi^0 = \mathscr{J} \setminus \bigcup_{\#T_j^{J'}(f(J')) \geq 3} \mathscr{J}^{J'}$. Let Ξ^1 be the set of all $J \in \Xi^0$ such that $J'' \in \mathscr{J}^{J_i}$ for some $J'' \in \bigcup_{\#T_j^{J'}(f(J')) \geq 3} \mathscr{J}^{J'}$ and a fixed $i \in \mathcal{V}$. Without loss of generality, we assume $J^{**} \in \Xi^1$ unless we have $\Xi^1 = \emptyset$. If $\Xi^1 \neq \emptyset$, let Ξ^2 be the set of all $J \in \Xi^0$ such that J_i^{**} is connected with J_i. If $\Xi^1 = \emptyset$, let $\Xi^2 = \emptyset$. Note that for all $J, J' \in \Xi^2$, J_i is connected with J_i'.

Construct the following CCR with f. For all $J \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^J$, if $J \in \mathscr{J} \setminus (\Xi^0 \cap \Xi^2)$, then $C_f(\mathbf{R}, f(J)) = G(f_E(J), R_i)$; and if $J \in \Xi^0 \cap \Xi^2$, then $C_f(\mathbf{R}, f(J)) = G(f_E(J), R_j)$ for a fixed $j \in \mathcal{V} \setminus \{i\}$. It is easy to show that C_f is well defined and satisfies weak Pareto, independence, and non-dictatorship from the assumptions of Lemma 7 and construction of C_f.

We now show that C_f satisfies nominee-efficient transitive-closure coherence. Let $B = \Xi^0 \cap \Xi^2$ and let $A = \mathscr{J} \setminus B$. We distinguish three possible subcases: (a) $\mathbf{R} \in \left(\bigcup_{J' \in A} \mathcal{D}^{J'}\right) \setminus \left(\bigcup_{J'' \in B} \mathcal{D}^{J''}\right)$; (b) $\mathbf{R} \in \left(\bigcup_{J' \in B} \mathcal{D}^{J'}\right) \setminus \left(\bigcup_{J'' \in A} \mathcal{D}^{J''}\right)$; and (c) $\mathbf{R} \in \left(\bigcup_{J' \in A} \mathcal{D}^{J'}\right) \cap \left(\bigcup_{J'' \in B} \mathcal{D}^{J''}\right)$.

Subcase (a). We have $C_f(\mathbf{R}, f(J)) = G(f_E(J), F(\mathbf{R}))$ by setting $F(\mathbf{R}) = R_i$ for all $\mathbf{R} \in \left(\bigcup_{J' \in A} \mathcal{D}^{J'}\right) \setminus \left(\bigcup_{J'' \in B} \mathcal{D}^{J''}\right)$. Therefore, it is NE-rationalized by a transitive-valued collective decision function F. It follows from Theorem 3 that C_f satisfies nominee-efficient transitive closure coherence in Case (a).

Subcase (b). We have $C_f(\mathbf{R}, f(J)) = G(f_E(J), F(\mathbf{R}))$ by setting $F(\mathbf{R}) = R_j$ for all $\mathbf{R} \in \left(\bigcup_{J' \in B} \mathcal{D}^{J'}\right) \setminus \left(\bigcup_{J'' \in A} \mathcal{D}^{J''}\right)$. Therefore, it is NE-rationalized by a transitive-valued collective decision function F. It follows from Theorem 3 that C_f satisfies nominee-efficient transitive closure coherence in Case (b).

Subcase (c). Suppose $\mathbf{R} \in \left(\bigcup_{J' \in A} \mathcal{D}^{J'}\right) \cap \left(\bigcup_{J'' \in B} \mathcal{D}^{J''}\right)$. Let $Q(\mathbf{R})$ be a binary relation over X as follows. For all $x, y \in X$, if $x, y \in T_i^{J'}(f(J'))$ for some $J' \in A$, then $(x, y) \in Q(\mathbf{R})$ if and only if $(x, y) \in R_i$; and if $x, y \in T_i^{J'}(f(J'))$ for some $J' \in B$, then $(x, y) \in Q(\mathbf{R})$ if and only if $(x, y) \in R_j$. $Q(\mathbf{R})$ is not always complete.

We now show that $Q(\mathbf{R})$ is well defined and transitive. First, for all $x, y \in X$, if there exist $J' \in A$ and $J'' \in B$ such that $x, y \in T_i^{J'}(f(J')) \cap T_i^{J''}(f(J''))$, then J_i' is directly related to J_i'', which contradicts the assumption that J_i' is not indirectly related to J_i''. Therefore, for all $x, y \in X$, all $J' \in A$, and all $J'' \in B$, if we have $x, y \in T_i^{J'}(f(J'))$, then we have $x \notin T_i^{J''}(f(J''))$ or $y \notin T_i^{J''}(f(J''))$, and vice versa. Thus, $Q(\mathbf{R})$ is well defined.

Second, for all $x, y, z \in X$, if there exist $J^1, J^2, J^3 \in A$ such that $x, y \in T_i^{J^1}(f(J^1))$, $y, z \in T_i^{J^2}(f(J^2))$, and $x, z \in T_i^{J^3}(f(J^3))$, then $Q(\mathbf{R})$ is transitive on $\{x, y, z\}$ from the construction of $Q(\mathbf{R})$. Similarly, for all $x, y, z \in X$, if there exist $J^1, J^2, J^3 \in B$ such that $x, y \in T_i^{J^1}(f(J^1))$, $y, z \in T_i^{J^2}(f(J^2))$, and

$x, z \in T_i^{J^3}(f(J^3))$, then $Q(\mathbf{R})$ is transitive on $\{x, y, z\}$ from the construction of $Q(\mathbf{R})$.

Suppose that for all $x^1, x^2, x^3 \in X$, there exist $J^3 \in A$ and $J^1, J^2 \in B$ such that $x^1, x^2 \in T_i^{J^1}(f(J^1))$, $x^2, x^3 \in T_i^{J^2}(f(J^2))$, and $x^1, x^3 \in T_i^{J^3}(f(J^3))$. Suppose $(x^1, x^2) \in Q(\mathbf{R})$ and $(x^2, x^3) \in Q(\mathbf{R})$, but $(x^3, x^1) \in P(Q(\mathbf{R}))$. From the construction of $Q(\mathbf{R})$, we have $(x^3, x^1) \in P(R_i)$. If $(x^1, x^2) \in R_i$, then there exists $R'_i \in \cap_{k\in\{1,2,3\}}\mathcal{D}_i^{J^k}$ such that $R_i|_{\{x^1,x^3\}} = R'_i|_{\{x^1,x^3\}}$, $x^3 \in G(T_i^{J^3}(f(J^3), R'_i)$, and $x^1 \in G(T_i^{J^1}(f(J^1), R'_i)$. This implies that J_i^3 is directly related to J_i^2, which is a contradiction. Therefore, if $R_i \in \cap_{k\in\{1,2,3\}}\mathcal{D}_i^{J^k}$, then $(x^2, x^1) \in P(R_i)$. Similarly, if $(x^2, x^3) \in R_i$, then there exist $R'_i, R''_i \in \cap_{k\in\{1,2,3\}}\mathcal{D}_i^{J^k}$ such that $(x^3, x^1) \in P(R'_i)$, $R'_i|_{\{x^1,x^3\}} = R''_i|_{\{x^1,x^3\}}$, $x^2 \in G(T_i^{J^2}(f(J^2), R''_i)$, and $x^3 \in G(T_i^{J^3}(f(J^3), R''_i)$. This implies that J_i^3 is directly related to J_i^1, which is a contradiction. Therefore, if $R_i \in \cap_{k\in\{1,2,3\}}\mathcal{D}_i^{J^k}$, then $(x^3, x^2) \in P(R_i)$. From the two cases, if $R_i \in \cap_{k\in\{1,2,3\}}\mathcal{D}_i^{J^k}$, then we must have $x^1, x^2, x^3 \in T_i^{J^3}(f(J^3))$. Suppose $\Xi^1 \neq \emptyset$. Then, for all $R'_i \in \cap_{k\in\{1,2\}}\mathcal{D}_i^{J^k}$, if we have $(x^2, x^1), (x^3, x^2) \in P(R'_i)$, then J_i^1 is not connected with J_i^2, which is a contradiction. Suppose $\Xi^1 = \emptyset$. Then, there exists $R'_i \in \cap_{k\in\{1,2,3\}}\mathcal{D}_i^{J^k}$ such that $(x^2, x^1) \in P(R'_i)$ and $(x^3, x^2) \in P(R'_i)$, $R_i|_{\{x^1,x^2\}} = R'_i|_{\{x^1,x^2\}}$ and $R_i|_{\{x^2,x^3\}} = R'_i|_{\{x^2,x^3\}}$, $x^3 \in G(T_i^{J^2}(f(J^2), R'_i)$, and $x^2 \in G(T_i^{J^1}(f(J^1), R'_i)$. This implies that both J_i^1 and J_i^2 are directly related to J_i^3, which is a contradiction. Thus, we have $(x^1, x^3) \in Q(\mathbf{R})$. However, if we have $(x^1, x^3) \in Q(\mathbf{R})$, then it is possible to show that we have a similar contradiction above. Therefore, for all $x^1, x^2, x^3 \in X$, there exist no $J^3 \in A$ and $J^1, J^2 \in B$ such that $x^1, x^2 \in T_i^{J^1}(f(J^1))$, $x^2, x^3 \in T_i^{J^2}(f(J^2))$, and $x^1, x^3 \in T_i^{J^3}(f(J^3))$.

Suppose that for all $x^1, x^2, x^3 \in X$, there exist $J^1, J^2 \in A$ and $J^3 \in B$ such that $x^1, x^2 \in T_i^{J^1}(f(J^1))$, $x^2, x^3 \in T_i^{J^2}(f(J^2))$, and $x^1, x^3 \in T_i^{J^3}(f(J^3))$. Suppose $(x^1, x^2) \in Q(\mathbf{R})$ and $(x^2, x^3) \in Q(\mathbf{R})$. From the construction of $Q(\mathbf{R})$, we have $(x^1, x^2), (x^2, x^3) \in R_i$, which implies $(x^1, x^3) \in R_i$. Then, there exist $R'_i, R''_i \in \cap_{k\in\{1,2,3\}}\mathcal{D}_i^{J^k}$ such that $(x^1, x^2) \in P(R'_i)$ and $(x^2, x^3) \in P(R'_i)$, $R'_i|_{\{x^1,x^2\}} = R''_i|_{\{x^1,x^2\}}$ and $R'_i|_{\{x^2,x^3\}} = R''_i|_{\{x^2,x^3\}}$, $x^1 \in G(T_i^{J^1}(f(J^1), R''_i)$, and $x^2 \in G(T_i^{J^2}(f(J^3), R''_i)$. This implies that J_i^1 and J_i^2 are directly related to J_i^3, which is a contradiction. Therefore, for all $x^1, x^2, x^3 \in X$, there exist no $J^1, J^2 \in A$ and $J^3 \in B$ such that $x^1, x^2 \in T_i^{J^1}(f(J^1))$, $x^2, x^3 \in T_i^{J^2}(f(J^2))$, and $x^1, x^3 \in T_i^{J^3}(f(J^3))$.

For any other cases, it is possible to show that $Q(\mathbf{R})$ is transitive by a similar argument above. Therefore, for all $J \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^J$, we have $C_f(\mathbf{R}, f(J)) = G(f_E(J), F(\mathbf{R}))$ by setting $F(\mathbf{R}) = Q(\mathbf{R})$. Thus, C_f is NE-rationalized by a transitive-valued collective decision function F. From Theorem 3, we can show that C_f satisfies nominee-efficient transitive closure coherence.

Case (ii). Let Ξ^3 be the set of all $J \in \mathscr{J}$ such that J_i^{**} is strongly connected with J_i. For all $J \in \mathscr{J}\backslash\Xi^3$ and all $J' \in \Xi^3$, J_i is not strongly connected with J'_i, and for all $J, J' \in \Xi^3$, J_i is strongly connected with J'_i.

Construct the following CCR with f. For all $J \in \mathscr{J}$ and all $\mathbf{R} \in \mathcal{D}^J$, if $J \in$

$\mathscr{J} \backslash \Xi^3$, then $C_f(\mathbf{R}, f(J)) = G(f_E(J), R_i)$; and if $J \in \Xi^3$, then $C_f(\mathbf{R}, f(J)) = G(f_E(J), R_j)$ for a fixed $j \in \mathcal{V}\backslash\{i\}$. It is easy to show that C_f is well defined and satisfies weak Pareto, independence, and non-dictatorship from the assumptions of Lemma 7 and construction of C_f. In addition, it is possible to show that C_f satisfies nominee-efficient transitive-closure coherence by a similar argument to Case (i). $\square$

Lemma 8. *Suppose that $T_i^J(f(J)) = T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$ and all $i, j \in \mathcal{V}$, and a CCR with f satisfies weak Pareto, independence, and nominee-efficient transitive-closure coherence. Then, for all $J, J' \in \mathscr{J}$, if J is directly related to J' and there exists a dictator d on $f(J)$, then d is a dictator on $f(J')$.*

Proof. Let d be a dictator on $f(J)$. We distinguish two possible cases: (i) $\#T_i^{J'}(f(J')) \geq 3$ or (ii) $\#T_i^{J'}(f(J')) < 3$.

Case (i). There exists a dictator d' on $f(J')$ because the preference domain $\mathcal{D}^{J'}$ violates the conditions of the Arrow-consistent preference domains in Theorem 1 in Iwata (2016). We show $d = d'$. Suppose $d \neq d'$. Since J is directly related to J', there exist alternatives $x, y \in T_j^{J'}(f(J'))$ and a preference profile $\mathbf{R} \in \mathcal{D}^{J'}$ such that $\{x\} = G(f(J'), R_d)$ and $\{y\} = G(f(J'), R_{d'})$ and there exist $K \in \mathbb{N}$, $x^0, \ldots, x^K \in X$, and $J^0, \ldots, J^K \in \mathscr{J}$ such that

1. $J' = J^0$, and either $J = J^1$ or $J = J^K$,
2. $x = x^0$ and $x^K = y$,
3. For all $i \in \mathcal{V}$, $x^{k-1}, x^k \in T_i^{J^k}(f(J^k))$ for all $k \in \{1, \ldots, K\}$,
4. There exists $\mathbf{R}' \in \cap_{k \in \{0,\ldots,K\}} \mathcal{D}^{J^k}$ such that $\{y\} = G(T_{d'}^{J^0}(f(J^0), R'_{d'})$ and for all $i \in \mathcal{V}$, either of the two conditions holds;
   (a) If $(x, y) \in P(R_i)$, then $\{x^{k-1}\} = G(T_i^{J^k}(f(J^k), R'_i)$ for all $k \in \{1, \ldots K\}$,
   or
   (b) If $(y, x) \in R_i$, then $\{x^{k-1}\} = G(T_i^{J^k}(f(J^k), R'_i)$ for all $k \in \{2, \ldots K\}$.

Suppose $J = J^1$. By the dictatorship of d on $f(J^1)$, we have $\{x\} = C_f(\mathbf{R}', f(J^1))$. By weak Pareto, we have $\{x^{k-1}\} = C_f(\mathbf{R}', f(J^k))$ for all $k \in \{2, \ldots, K\}$. Therefore, we have $(x^{k-1}, x^k) \in R^E_{C_f^{\mathbf{R}'}}$ for all $k \in \{1, \ldots, K\}$, which implies that $(x, y) \in tc(R^E_{C_f^{\mathbf{R}'}})$. Since C_f satisfies nominee-efficient transitive-closure coherence, it is NE-rationalized by a transitive-valued collective decision function F from Theorem 3. Therefore, we have $tc(R^E_{C_f^{\mathbf{R}'}}) \subseteq F(\mathbf{R}')$ from Lemma 2, which implies that $(x, y) \in F(\mathbf{R}')$. Since C_f is NE-rationalized by F, $y \in C_f(\mathbf{R}', f(J^0))$ implies $x \in C_f(\mathbf{R}', f(J^0))$. However, by the dictatorship of d' on $f(J^0)$, we have $\{y\} = C_f(\mathbf{R}', f(J^0))$, which is a contradiction. Thus, d is a dictator on $f(J')$.

Suppose $J = J^K$. Then, it is possible to show that d is a dictator on $f(J')$ by a similar argument above.

Case (ii). Suppose that d is not a dictator on $f(J')$. Then, there exist $x, y \in f(J')$ and $\mathbf{R} \in \mathcal{D}^{J'}$ such that $(x, y) \in P(R_d)$ and $y \in C_f(\mathbf{R}, f(J'))$, which implies that $\{x, y\} = T_j^{J'}(f(J'))$ for all $j \in \mathcal{V}$.

Since J is directly related to J', there exist $K \in \mathbb{N}$, $x^0, \ldots, x^K \in X$, and $J^0, \ldots, J^K \in \mathscr{J}$ such that

1. $J' = J^0$, and either $J = J^1$ or $J = J^K$,
2. $x = x^0$ and $x^K = y$,
3. For all $i \in \mathcal{V}$, $x^{k-1}, x^k \in T_i^{J^k}(f(J^k))$ for all $k \in \{1, \ldots, K\}$,
4. There exists $\mathbf{R}' \in \cap_{k \in \{0,\ldots,K\}} \mathcal{D}^{J^k}$ such that $\mathbf{R}|_{\{x,y\}} = \mathbf{R}'|_{\{x,y\}}$ and for all $i \in \mathcal{V}$, either of the two conditions holds;
   (a) If $(x, y) \in P(R_i)$, then $\{x^{k-1}\} = G(T_i^{J^k}(f(J^k), R_i')$ for all $k \in \{1, \ldots K\}$,
   or
   (b) If $(y, x) \in R_i$, then $\{x^{k-1}\} = G(T_i^{J^k}(f(J^k), R_i')$ for all $k \in \{2, \ldots K\}$.

Suppose $J = J^1$. If we have $y \notin C_f(\mathbf{R}', f(J^0))$, then $x \in C_f(\mathbf{R}', f(J^0))$ must hold. However, this contradicts independence. Therefore, we have $y \in C_f(\mathbf{R}', f(J^0))$, which implies that $(y, x) \in R^E_{C_f^{\mathbf{R}'}}$. By weak Pareto, we have $\{x^{k-1}\} = C_f(\mathbf{R}', f(J^k))$ for all $k \in \{2, \ldots, K\}$. Therefore, we have $(x^{k-1}, x^k) \in R^E_{C_f^{\mathbf{R}'}}$ for all $k \in \{2, \ldots, K\}$, which implies that $(x^1, x) \in tc(R^E_{C_f^{\mathbf{R}'}})$. Since C_f satisfies nominee-efficient transitive-closure coherence, it is NE-rationalized by a transitive-valued collective decision function F from Theorem 3. Therefore, we have $tc(R^E_{C_f^{\mathbf{R}'}}) \subseteq F(\mathbf{R}')$ from Lemma 2, which implies that $(x^1, x) \in F(\mathbf{R}')$. Since C_f is NE-rationalized by F, $x \in C_f(\mathbf{R}', f(J^1))$ implies $x^1 \in C_f(\mathbf{R}', f(J^1))$. However, by the dictatorship of d on $f(J^1)$, we have $\{x\} = C_f(\mathbf{R}', f(J^1))$, which is a contradiction. Thus, d is a dictator on $f(J')$.

Suppose $J = J^K$. Then, it is possible to show that d is a dictator on $f(J')$ by a similar argument above. □

Lemma 9. *Suppose that $T_i^J(f(J)) = T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$ and all $i, j \in \mathcal{V}$, and a CCR with f satisfies weak Pareto, independence, and nominee-efficient transitive-closure coherence. Then, for all $J, J' \in \mathscr{J}$, if J is indirectly related to J' and there exists a dictator d on $f(J)$, then d is a dictator on $f(J')$.*

Proof. Let d be a dictator on $f(J)$. Since J is indirectly related to J', there exist $K \in \mathbb{N}$ and $J^0, \ldots, J^K \in \mathscr{J}$ such that $J = J^0$, $J^K = J'$, and J^{k-1} is directly related to J^k for all $k \in \{1, \ldots, K\}$. From Lemma 8, d is a dictator on $f(J^1)$. By using Lemma 8 repeatedly, for all $k \in \{2, \ldots, K\}$, if d is a dictator on $f(J^{k-1})$, then d is a dictator on $f(J^k)$, which implies that d is a dictator on $f(J')$. □

We are now ready to prove Theorem 4.

Proof of Theorem 4. "Only-if" part: Suppose that either condition 1 or condition 2 fails. If condition 1 fails, then there exist $J \in \mathscr{J}$ and $x \in f(J)$ such that $x \in T_j^J(f(J))$ for all $j \in \mathcal{V}$ and $x \notin f_E(J)$. Therefore, there exists no CCR with f satisfying weak Pareto and nominee-efficient transitive closure coherence from Lemma 3. If condition 2 fails, then $T_i^J(f(J)) = T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$ and all $i, j \in \mathcal{V}$, and either of the following two cases holds: (i) for all $J \in \mathscr{J}$, $\#T_j^J(f(J)) < 2$ for all $j \in \mathcal{V}$; or (ii) for all $J \in \mathscr{J}$ with $\#T_j^J(f(J)) \geq 3$ for all $j \in \mathcal{V}$, J is strongly connected with all $J' \in \mathscr{J}$ with $\#T_j^{J'}(f(J')) \geq 2$ for all $j \in \mathcal{V}$.

If Case (i) holds, then any CCR with f violates non-dictatorship or weak Pareto is not compatible with non-dictatorship. Suppose that Case (ii) holds. Since we have $\#T_j^J(f(J)) \geq 3$, there exists a dictator d on $f(J)$ because the preference domain $\mathcal{D}^J$ violates the conditions of the Arrow-consistent preference domains in Theorem 1 in Iwata (2016). We now show that d is a dictator on $f(J')$ for all $J' \in \mathscr{J}$. Since J is strongly connected with J' from the assumption of Case (ii), there exist $K \in \mathbb{N}$ and $\mathscr{J}^{J^0}, \ldots, \mathscr{J}^{J^K} \subseteq \mathscr{J}$ with $\#T_j^{J^k}(f(J^k)) \geq 3$ for all $j \in \mathcal{V}$ and all $k \in \{1, \ldots, K\}$ such that $J \in \mathscr{J}^{J^0}$, $J' \in \mathscr{J}^{J^K}$, and $\mathscr{J}^{J^{k-1}} \cap \mathscr{J}^{J^k} \neq \emptyset$ for all $k \in \{1, \ldots, K\}$.

There exists a dictator d_k on $f(J^k)$ for all $k \in \{1, \ldots, K\}$ because the preference domain $\mathcal{D}^{J^k}$ violates the conditions of Arrow-consistent preference domains in Theorem 1 in Iwata (2016). From Lemma 9, for all $k \in \{1, \ldots, K\}$, d_k is a dictator on $f(J'')$ for all $J'' \in \mathscr{J}^{J^k}$.

Since we have $\mathscr{J}^{J^0} \cap \mathscr{J}^{J^1} \neq \emptyset$, there exists $J'' \in \mathscr{J}^{J^0} \cap \mathscr{J}^{J^1}$. Therefore, if $(d =)d_0 \neq d_1$, then we have a contradiction. Thus, we have $d_0 = d_1$. By repeating this argument, we have $d_{k-1} = d_k$ for all $k \in \{1, \ldots, K\}$, which implies that d is a dictator on $f(J')$.

"If" part: Suppose that conditions 1 and 2 of Theorem 4 hold. From condition 1, we have $\cap_{j \in \mathcal{V}} T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$. If condition 2 (a) holds, then there exists a CCR with f satisfying weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence from Lemma 4. If condition 2 (b) holds, then there exists a CCR with f satisfying weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence from Lemma 5. If conditions 2 (a) and 2 (b) fail, then we have $T_i^J(f(J)) = T_j^J(f(J)) \subseteq f_E(J)$ for all $J \in \mathscr{J}$ and all $i, j \in \mathcal{V}$. If condition 2 (c) (respectively, condition 2 (d)) holds, then there exists a CCR with f satisfying weak Pareto, independence, non-dictatorship, and nominee-efficient transitive-closure coherence from Lemma 6 (respectively, Lemma 7). □

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