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Abstract

In this study, we extend the inequality measure of power in single-issue voting situations, which was originally proposed by Laruelle and Valenciano (Social Choice and Welfare 22: 413–431, 2004). In our model, there exist multiple issues on which to vote, and for each issue, each voter has one of three voting options—namely, "yes," "no," or "abstention." We add a separability axiom to the set of Laruelle and Valenciano's axioms that characterize their inequality measure of voting power. Our inequality measure is represented by a two-stage aggregation procedure, where in the first stage, the distribution of voting power with regards to each issue is aggregated into an inequality index based on the Laruelle–Valenciano measure; then, the second-stage aggregator sums up the inequality index on each issue.

JEL classifications: D63, D72

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1 Introduction

In studies of power in voting situations, assessments of the various distributions of voting power have received less attention than measurements of the distribution of voting power. The distribution of voting power has been numerically represented by various power indices, such as the Banzhaf (1965) and Shapley and Shubik (1954) indices. On the other hand, only a few studies—including, for example, those of Einy and Peleg (1991) and Laruelle and Valenciano (2004)—propose ways of assessing various distributions of voting power.[1] Einy and Peleg (1991) axiomatize a family of inequality measurements of voting power distributions for cooperative games (TU-games), which are generalized

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[1] Weber (2016) recently considered an inequality measurement of two-tier voting systems, in which a binary decision is made by the representatives of different groups, and the members of each group determine the vote of the representative.

Gini functions on distributions based on the Shapley–Shubik index.[2] Laruelle and Valenciano (2004) point out that Einy and Peleg (1991) arbitrarily adopt the Shapley–Shubik index as a voting power measurement. They do not restrict their attention to any particular power index, and consider a broader conceptual framework where voting power is defined as the probability of playing a crucial role in collective decision-making.[3] They construct their inequality measure of voting power by imposing some reasonable properties on the class of inequality measures.

In the current study, we extend the normative implication and the practical application of the Laruelle and Valenciano (2004) inequality measure of voting power. They had originally intended to develop an inequality measure of voting power in real-world collective decision-making situations.[4] Nevertheless, their model does not capture some realistic situations, because the model is based on simple games, where it is assumed that voters cast one of two votes—namely, "yes" or "no"—to determine the passage or the rejection of a single issue. Felsenthal and Machover (1997) point out that the "yes" or "no" votes of voters lack in reality from a practical viewpoint, and propose ternary voting games, where voters have a third option—namely, "abstention"—in addition to "yes" and "no" options.[5] We adopt ternary voting games and extend the Laruelle–Valenciano concept of voting power to such games.

We believe that the second, and more important, lack of reality in simple games is that the set of issues on which to vote is a singleton. This problem has attracted little attention in the literature on the measurement of voting power. In real-world voting situations, different voting rules are often applied to different issues: decisions of the UN Security Council on procedural matters, for example, are made by the "yes" votes of nine members, while its decisions on nonprocedural matters are made by the "yes" votes of nine members including all permanent members. Thus, we can consider voting situations that involve multiple issues. The aim of this study is to measure the overall inequality of voting powers when there exist multiple issues on which a collective decision is to be made.

Our definition of "voting power" is compatible with the concept of the system of powers proposed by Ju (2010). He considers an opinion aggregation problem where all members in a society have positive, negative, or neutral opinions on each of a number of issues. Society aggregates their opinions and makes a binary decision—namely, acceptance or rejection—on each issue. Our model is relevant

[2]In the literature on inequality of income distributions, generalized Gini functions are axiomatically characterized by Waymark (1981).

[3]This general framework is proposed by Laruelle and Valenciano (2005).

[4]See Aleskerov et al. (2000), Beisbart and Bovens (2008), and Laruelle and Valenciano (2002) for practical applications of the Laruelle–Valenciano inequality measure of voting power.

[5]Simple games are extended to (j, k) simple games by Bolger (1993), where j is the number of possible options to vote and k is the number of possible outcomes in a voting situation, and both options and outcomes are not always ordered in a natural way. Then, simple games are $(2, 2)$ simple games and ternary voting games are $(3, 2)$ simple games, but both options and outcomes can be interpreted as being naturally ordered.

to that of Ju (2010) when "abstention" in our model is interpreted as a neutral opinion in his model. Ju (2010) introduces a system of powers to represent an individual's conditional decisive power with regards to each issue; the system is a function that assigns each issue to an individual who has power with regards to the issue and the minimum social consent quotas needed for that individual to influence social decisions. Ju (2010) characterizes the class of decision rules that are represented by a system of powers, with three axioms; these axioms include a *monotonicity* axiom and an *independence* axiom.[6] Since we impose no restrictions on decision rules, the class of decision rules in our model contains those that are represented by Ju's (2010) system of powers.

Our question is how to measure the overall inequality of voting powers when multiple issues on which to vote exist. To answer this question, we propose a two-stage aggregation procedure. In the first stage, the distributions of voting power with regards to each issue are aggregated by using an inequality index based on the Laruelle–Valenciano measure; then, the second-stage aggregator sums up the inequality index on each issue. Laruelle and Valenciano (2004) provide axiomatic characterization of their inequality measure of voting power in single-issue voting situations, through some reasonable properties. We add a *separability* axiom to the set of their axioms to characterize our inequality measure of voting powers in multiple-issue voting situations.

Furthermore, we consider the case where the number of voters and the number of issues are variable. We propose three equivalence principles with respect to the number of voters or the number of issues, two of which are similar to those of Laruelle and Valenciano (2004). As a consequence, we show that our inequality measure of voting powers can be seen as the arithmetic mean of the Laruelle–Valenciano inequality index of voting power with regards to each issue.

The remaining sections are organized as follows. Section 2 introduces the model and basic notation. Section 3 defines the concept of voting powers, as seen in our model. In Section 4, we extend the domain of power profiles to lotteries over decision rules. Section 5 characterizes our inequality index for power profiles. Section 6 extends the model to the case where the number of voters and the number of issues are variable. Section 7 summarizes the paper.

2 Decision Rules

A simple game is a general procedure to make a collective decision on a single issue, by the "yes" or "no" votes of the members of a committee. We extend simple games in two directions. In one, the number of issues on which to vote is multiple, rather than single; in the other, each member has three voting options—"yes," "no," or "abstention"—on each issue.

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ denote the set of all natural numbers. Let $N = \{1, \ldots, n\}$ be the set of n *voters* with $n > 1$. Let $M =$

[6] The third axiom introduced by Ju (2010) is a symmetry condition, that under at least one linkage from issues to individuals, the rule should symmetrically treat individual i and issues linked to i and any other individual i' and issues linked to i'.

$\{1, \ldots, m\}$ be the set of m *issues* with $m \geq 1$. It is assumed that each voter votes for or against each issue, and can also abstain from voting on some issues. A voter i's vote is represented by an $1 \times m$ row vector T_i consisting of -1, 0, or 1. Let T_{ij} be the jth component of T_i; the interpretation is that $T_{ij} = 1$ when i is voting for issue j, and $T_{ij} = -1$ when i is voting against. If $T_{ij} = 0$, i abstains from voting on issue j. A *voting profile* is a possible list of voting behaviors by the voters and it is denoted by an $n \times m$ matrix T consisting of n row vectors $T_1, \ldots, T_n$. Thus, we have 3^{nm} possible voting profiles. Let $\mathscr{T}_{nm}$ be the set of all voting profiles when n voters and m issues exist. Let (T'_i, T_{-i}) be the voting profile obtained by replacing T_i of T with T'_i.

A *decision rule* for n voters and m issues specifies which voting profiles will lead to the passage of which issues, and which ones will lead to their rejection. We shall represent a decision rule with a subset of voting profiles. When a voting profile leads to the passage of an issue j, it is called a *winning profile for* j. Let $\mathscr{W}_j$ denote the set of winning profiles for j. Let $\mathscr{W} = (\mathscr{W}_1, \ldots, \mathscr{W}_m)$ be a list of winning profiles for any issue j, which will represent a decision rule for n voters and m issues. A decision rule is *trivial* for any issue j, if either $T \notin \mathscr{W}_j$ or $T \in \mathscr{W}_j$ holds for any $T \in \mathscr{T}_{nm}$. Thus, we allow that some issues may always pass or fail.

Let $\mathcal{D}_{nm}$ be the set of all such decision rules for n voters and m issues. Each of them identifies with the list $\mathscr{W}$ of winning profiles for any issue j. We will prove our results by using some specific decision rules. A decision rule is voter i's *dictatorship* on any issue j if her vote is decisive on j, in the sense that issue j is passed if and only if she votes "yes" for j—that is, for any $T \in \mathscr{T}_{nm}$ and any $j \in M$, $T \in \mathscr{W}_j$ if and only if $T_{ij} = 1$. A decision rule is voter i's *simple dictatorship* on any issue j, if it is voter i's dictatorship on issue j and is trivial for any other issues j'. Let $d_j^{\{i\}}$ denote voter i's simple dictatorship on j. A voter i is a *null-voter* for any issue j in a decision rule, if her voting behavior does not influence decision on j—that is, for any $T \in \mathscr{T}_{nm}$, $T \in \mathscr{W}_j$ if and only if $(T'_i, T_{-i}) \in \mathscr{W}_j$ for any T'_i.

3 A Measurement of Voting Powers

As in Laruelle and Valenciano (2004, 2005), we define *a priori* voting power with regards to any issue j as the probability of voting profiles in which a voter i exerts power for j, in the sense that she plays a crucial role in making a collective decision on j according to a given decision rule. As we can see below, the *a priori* voting power of any voter i is represented by a vector, each of whose components denotes the probability of that voter exerting power with regards to any issue j.

We first consider the definition of exerting voting powers in a reasonable subclass of decision rules. Thereafter, we provide the general definition of exerting voting powers. Let us assume that a decision rule is *monotonic* and *independent.*[7] Monotonicity requires that a decision on any issue j should not respond

[7] Ju (2010, Proposition 5) shows that a decision rule satisfies monotonicity and indepen-

negatively whenever any voting profile increases in the sense of the matrix. Formally, for any $T, T' \in \mathscr{T}_{nm}$ with $T \leq T'$ and any $j \in M$, if $T \in \mathscr{W}_j$, then $T' \in \mathscr{W}_j$. Independence requires that the decision on any issue j depends solely on the votes of voters for j. Formally, for any $T, T' \in \mathscr{T}_{nm}$ and any $j \in M$, if $T_{ij} = T'_{ij}$ for any $i \in N$, then $T \in \mathscr{W}_j$ if and only if $T' \in \mathscr{W}_j$. Let $\hat{\mathcal{D}}_{nm}$ be the subclass of decision rules that satisfy monotonicity and independence.

If a decision rule is monotonic, then the decision on j does not respond negatively (positively) when any voter i changes her votes in a positive (negative) way. In addition, if a decision rule is independent, the decision on each issue is independent of any other issue. Suppose that any voter except for i does not change her vote. Then, a voter i exerts positive (negative) power for any issue j if (i) she votes "yes" ("no") on j or abstains from voting on j, (ii) the issue j is passed (rejected), and (iii) the decision responds in the opposite manner when she changes her votes in a negative (positive) way. In general, we can say that the voter exerts power in positive and negative senses, or both. That is, her vote on j is crucial to the decision on j. These notions of voting power are natural extensions of those of Laruelle and Valenciano (2004, 2005) in the "yes" and "no" options case.

Formally, given a decision rule $\mathscr{W} \in \hat{\mathcal{D}}_{nm}$, we first introduce the following notions of voting power:

$$\hat{\mathscr{E}}^{+}_{ij} = \{T \in \mathscr{T}_{nm} : \text{There exists } T'_i \text{ with } T_{ij} \geq T'_{ij} \text{ such that } T \in \mathscr{W}_j \text{ and } (T'_i, T_{-i}) \notin \mathscr{W}_j\}$$

and

$$\hat{\mathscr{E}}^{-}_{ij} = \{T \in \mathscr{T}_{nm} : \text{There exists } T'_i \text{ with } T'_{ij} \geq T_{ij} \text{ such that } T \notin \mathscr{W}_j \text{ and } (T'_i, T_{-i}) \in \mathscr{W}_j\}.$$

Thus, $\hat{\mathscr{E}}^{+}_{ij}$ is the set of all voting profiles such that the decision on j changes from acceptance to rejection when voter i changes her votes on j in a negative manner. On the other hand, $\hat{\mathscr{E}}^{-}_{ij}$ is the set of all voting profiles such that the decision on j changes from rejection to acceptance when the voter changes her votes on j in a positive manner. As mentioned in the Introduction, our model is relevant to that of Ju (2010) when "abstention" in our model is considered synonymous with his model's concept of "neutral opinion." Example 1 illustrates the relationship between our notion of power and that of Ju (2010).

Example 1. Consider the model of a system of powers proposed by Ju (2010). To define Ju's (2010) concept of power, we need *consent quotas* that represent the degrees of social consent required for the exercise of the voter's power. In this example, we consider a specific case. Given a decision rule $\mathscr{W} \in \hat{\mathcal{D}}_{nm}$, for any voter i and any issue j, suppose that the decision on j is made as follows. For any $T \in \mathscr{T}_{nm}$,

$$\begin{aligned} &\text{when } T_{ij} = 1, T \in \mathscr{W}_j \Leftrightarrow \#\{i' \in N : T_{i'j} = 1\} \geq n; \\ &\text{when } T_{ij} = 0, T \in \mathscr{W}_j \Leftrightarrow \#\{i' \in N : T_{i'j} = 1\} \geq n; \\ &\text{when } T_{ij} = -1, T \notin \mathscr{W}_j \Leftrightarrow \#\{i' \in N : T_{i'j} = -1\} \geq 1, \end{aligned} \qquad (1)$$

dence if and only if it is represented by a list of "decisive structures" that comprise the pair of disjoint subsets of N with a monotonic property in terms of their set inclusion.

where # denotes the cardinality of a set.

The last parts of the three equations in Equation (1) denote consent quotas. Then, according to Ju's (2010) definition, voter i has the power with regards to issue j. Under the decision rule, voter i requires unanimous consent to accept issue j, while she does not need any consent to reject issue j. Thus, Ju (2010) defines power as conditional (on social consent) decisiveness, which is a concept that relates to decision rules. Under the decision rule, note that the decision on j does not change for any voting profile when voter i is replaced with any other voter i'.

On the other hand, our concept of power with regards to j relates to the decision on j for two voting profiles, in which only i votes in a different manner. In our terminology, every voter i exerts power with regards to issue j in both positive and negative senses only when any other voter votes for j. That is, any voting profile $T \in \mathscr{T}_{nm}$ such that $T_{ij} = 1$ for any $i \in N$ is contained in $\hat{\mathscr{E}}_{ij}^{+}$, and any voting profile $T \in \mathscr{T}_{nm}$ such that $T_{ij} \leq 0$ and $T_{i'j} = 1$ for any $i' \in N \backslash \{i\}$ is contained in $\hat{\mathscr{E}}_{ij}^{-}$. Thus, in this case, the power with regards to j is equally shared by all voters. In general, if a decision rule is represented by Ju's (2010) system of powers, then we can define the two sets $\hat{\mathscr{E}}_{ij}^{+}$ and $\hat{\mathscr{E}}_{ij}^{-}$ according to the decision rule.

We now provide the general definition of exerting voting powers under any decision rule. Since a decision rule is not always monotonic, the decision on an issue j could respond negatively when a voting profile increases in the sense of the matrix. Moreover, the decision rule is not always independent; the decision on an issue j could depend on the votes of voters for any other issue j'. Therefore, we define that a voter i exerts positive (negative) power for an issue j if (i) the issue j is passed (rejected) and (ii) the decision responds oppositely when the voter changes her votes in any manner. Thus, we can say that she exerts voting power for j in the general sense when her vote (not only on issue j) is crucial to the decision on j.

Formally, given a decision rule $\mathscr{W} \in \mathcal{D}_{nm}$, we extend the aforementioned notions of voting power to the case where the decision rule $\mathscr{W}$ is not always monotonic and independent:

$$\mathscr{E}_{ij}^{+} = \{T \in \mathscr{T}_{nm} : \text{There exists } T_i' \text{ such that } T \in \mathscr{W}_j \text{ and } (T_i', T_{-i}) \notin \mathscr{W}_j\}$$

and

$$\mathscr{E}_{ij}^{-} = \{T \in \mathscr{T}_{nm} : \text{There exists } T_i' \text{ such that } T \notin \mathscr{W}_j \text{ and } (T_i', T_{-i}) \in \mathscr{W}_j\}.$$

In what follows, we adopt the general notions of voting power, in order to measure the inequality of voting powers.

Those aforementioned definitions relate to *a posteriori* descriptions of voting powers after the vote is cast. The problem is how to evaluate the *a priori* voting powers of various voters in our voting situations, where "*a priori*" means "prior to the vote being cast." As Laruelle and Valenciano (2004) point out, *a priori* voting powers in general depend on the structure of the decision rule and the probability over voting profiles. Clearly, the decision rule $\mathscr{W}$ influences whether a voting profile T is contained in $\mathscr{E}_{ij}^{+}$ or $\mathscr{E}_{ij}^{-}$. From the viewpoint of evaluating *a priori* voting powers, the probability of T influences the voters' expectations of exerting powers. Thus, we use a distribution of probability over all conceivable voting profiles to evaluate the *a priori* voting powers of various voters. Let $p : \mathscr{T}_{nm} \to \mathbb{R}$ be a probability distribution over all possible voting profiles—that is, a mapping that associates with each voting profile $T \in \mathscr{T}_{nm}$ its probability of occurrence $p(T)$, where $0 \leq p(T) \leq 1$ for any $T \in \mathscr{T}_{nm}$, and $\sum_{T\in\mathscr{T}_{nm}} p(T) = 1$. Let $\mathscr{P}_{nm}$ be the set of all distributions of probability over $\mathscr{T}_{nm}$.

Given a decision rule and a distribution of probability over all voting profiles, voter i's *a priori* voting power with regards to issue j can be defined as the probability of voting profiles such that voter i can exert power with regards to issue j. Formally, we define the following notion.

Definition 1. *For a given decision rule $\mathscr{W} \in \mathcal{D}_{nm}$ and a distribution of probability over the voting profiles $p \in \mathscr{P}_{nm}$, voter i's power with regards to issue j is given by:*

$$\Phi_i^j(\mathscr{W}, p) := \sum_{T\in\mathscr{E}_{ij}^{+}} p(T) + \sum_{T\in\mathscr{E}_{ij}^{-}} p(T) = \sum_{\substack{T\in\mathscr{E}_{ij}^{+} \\ (T_i', T_{-i})\in\mathscr{E}_{ij}^{-}}} (p(T) + p(T_i', T_{-i})). \quad (2)$$

Let $\Phi_i(\mathscr{W}, p)$ be a list of voter i's voting powers with regards to any issue j—that is, we define $\Phi_i(\mathscr{W}, p) := (\Phi_i^1(\mathscr{W}, p), \ldots, \Phi_i^m(\mathscr{W}, p))$. Thus, voter i's voting powers are represented by a vector, each of whose components is the voter's voting power with regards to any issue j. Let $\Phi(\mathscr{W}, p)$ be a list of all voters' voting powers, called a *power profile*. That is, we define

$$\Phi(\mathscr{W}, p) = \begin{pmatrix} \Phi_1^1(\mathscr{W}, p) & \cdots & \Phi_1^m(\mathscr{W}, p) \\ \vdots & \ddots & \vdots \\ \Phi_n^1(\mathscr{W}, p) & \cdots & \Phi_n^m(\mathscr{W}, p) \end{pmatrix} = (\Phi_1(\mathscr{W}, p), \ldots, \Phi_n(\mathscr{W}, p))^{-1}.$$

4 Extended Power Profiles

In this section, we extend power profiles to profiles with probabilities over decision rules, in which voter i's power with regards to issue j is represented by the probability of her exerting power when a decision rule $\mathscr{W}$ and a voting profile T are randomly chosen according to their probabilities. As Laruelle and Valenciano (2004) explain, any inequality index on the set of power profiles—that is, the set $\{\Phi(\mathscr{W}, p) : \mathscr{W} \in \mathcal{D}_{nm}\}$ for any $p \in \mathscr{P}_{nm}$—would be ordinally equivalent to any increasing monotonic transformation of it. In addition, Laruelle and Valenciano (2004) assume that a normative preference over power profiles is a von Neumann–Morgenstern preference on some subsets of the profiles, used to restrict the degrees of freedom for an inequality index.

Let us introduce some notation to define the probability profiles of voting powers. Let $\mathscr{L}(\mathcal{D}_{nm})$ denote lotteries over the set of decision rules. A *lottery*

$l \in \mathscr{L}(\mathcal{D}_{nm})$ can be represented by the mapping $l : \mathcal{D}_{nm} \to \mathbb{R}$, such that for any $\mathscr{W} \in \mathcal{D}_{nm}$, $l(\mathscr{W}) \geq 0$, and $\sum_{\mathscr{W} \in \mathcal{D}_{nm}} l(\mathscr{W}) = 1$, where $l(\mathscr{W})$ is the probability of making a collective decision according to decision rule $\mathscr{W}$. We consider $\mathcal{D}_{nm} \subseteq \mathscr{L}(\mathcal{D}_{nm})$, which implies that each decision rule $\mathscr{W}$ could be chosen with probability 1 (i.e., $l(\mathscr{W}) = 1$). Given $l, l' \in \mathscr{L}(\mathcal{D}_{nm})$, and $\lambda \in [0, 1]$, let $\lambda l \oplus (1 - \lambda)l'$ denote the lottery such that $(\lambda l \oplus (1 - \lambda)l')(\mathscr{W}) := \lambda l(\mathscr{W}) + (1 - \lambda)l'(\mathscr{W})$. We define any "convex combination" of lotteries in a similar way. For example, $\frac{1}{2}\mathscr{W} \oplus \frac{1}{2}\mathscr{W}'$ will be the lottery that assigns one half to decision rule $\mathscr{W}$ and the other half to $\mathscr{W}'$. The *support* of a lottery l is the set $\sup(l) := \{\mathscr{W} \in \mathcal{D}_{nm} : l(\mathscr{W}) > 0\}$.

The general measure of voter i's voting power with regards to issue j, which is given by Equation (2), can be naturally extended by using lotteries in $\mathscr{L}(\mathcal{D}_{nm})$. For any lottery $l \in \mathscr{L}(\mathcal{D}_{nm})$, and any probability distribution $p \in \mathscr{P}_{nm}$, we define

$$\bar{\Phi}_i^j(l, p) := \sum_{\mathscr{W} \in \mathcal{D}_{nm}} l(\mathscr{W})\Phi_i^j(\mathscr{W}, p),$$

which represents the probability of voter i exerting power with regards to issue j when decision rule $\mathscr{W}$ is randomly chosen according to lottery l, and $p(T)$ is the probability of each voting profile $T \in \mathscr{T}_{nm}$. Let $\bar{\Phi}_i(l, p) = (\bar{\Phi}_i^1(l, p), \ldots, \bar{\Phi}_i^m(l, p))$ be a list of such extended voter i's powers with regards to any issue j. We now define an *extended power profile* $\bar{\Phi}(l, p)$ as follows:

$$\bar{\Phi}(l, p) = \begin{pmatrix} \bar{\Phi}_1^1(l, p) & \cdots & \bar{\Phi}_1^m(l, p) \\ \vdots & \ddots & \vdots \\ \bar{\Phi}_n^1(l, p) & \cdots & \bar{\Phi}_n^m(l, p) \end{pmatrix} = (\bar{\Phi}_1(l, p), \ldots, \bar{\Phi}_n(l, p))^{-1}.$$

Thus, the domain of extended power profiles composes the set $\bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm})$, or $\cup_{p \in \mathscr{P}_{nm}} \bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm} \times \{p\})$, of $n \times m$ matrices. We construct an inequality index on this domain. Note that those $n \times m$ matrices can be interpreted as nm-dimensional vectors in $\mathbb{R}^{nm}$. In the next section, we will interpret extended power profiles as nm-dimensional vectors, rather than $n \times m$ matrices. In addition, as Laruelle and Valenciano (2004) mention, the properties to single out an inequality index are applicable to any restricted domains of extended power profiles whose form is $\bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \{p\})$ for any $p \in \mathscr{P}_{nm}$.

5 An Inequality Index for Extended Power Profiles

In this section, we construct an inequality index for extended power profiles that represent the probabilities of exerting powers, in the sense that voters play a crucial role in making a collective decision. In summary, we propose the following composition in voting situations for n voters and m issues:

$$\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm} \xrightarrow{\bar{\Phi}} \bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm}) \xrightarrow{I_{nm}} \mathbb{R}.$$

A function I_{nm} assigns a real number to each extended power profile in $\bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm})\times \mathscr{P}_{nm})$, which is interpreted as a measure of the degree of inequality in voting powers among various voters. After all, we consider a composite index $I_{nm} \circ \bar{\Phi}$ that compares all pairs (l, p) in $\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm}$. If a probability p over voting profiles is fixed, it compares extended power profiles under the fixed probability p.

As in Laruelle and Valenciano (2004), we intend to single out an inequality index by successively imposing some reasonable properties on such indices; this will gradually narrow down the class of them. As mentioned in Section 4, any extended power profile $\bar{\Phi}(l, p)$ is represented by an $n \times m$ matrix, while it can be interpreted as an nm-dimensional vector in $\mathbb{R}^{nm}$. Let $\alpha = (\alpha^1, \ldots, \alpha^m)$ be an nm-dimensional vector, where α^j is an n-dimensional vector for any $j \in \{1, \ldots, m\}$ and corresponds to the jth column vector of the $n \times m$ matrix α. Let α_i^j be the ith component of α^j. To focus on α^j, we may write $\alpha = (\alpha^j, \alpha^{-j})$. Therefore, (β^j, α^{-j}) is the nm-dimensional vector obtained by replacing α^j of α with β^j.

We first consider only the extended power profiles in the $(nm-1)$-simplex Δ_{nm}, whose extreme points are the canonical basis $\{e_i^j : i \in N, j \in M\}$ of $\mathbb{R}^{nm}$, where

$$e_{i'}^{j'} = \begin{cases} 1 & \text{if } i = i' \text{ and } j = j' \\ 0 & \text{if otherwise.} \end{cases}$$

Proposition 1 shows that any subdomain of an inequality index, which is $\bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \{p\})$ for any $p \in \mathscr{P}_{nm}$, contains the simplex Δ_{nm}. In the next step, we characterize an inequality index on the simplex Δ_{nm} and extend it to the whole domain $\bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm})$ of all extended power profiles.

Proposition 1. *For any $p \in \mathscr{P}_{nm}$, $\Delta_{nm} \subseteq \bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \{p\})$.*

Proof. We will show that extended power profiles in Δ_{nm} can be constructed by using the lotteries over the nm possible simple dictatorships, irrespective of the distribution of probability over the voting profiles. Note that for any $i \in N$ and any $j \in M$, regardless of the distribution $p \in \mathscr{P}_{nm}$, it follows from Equation (2) that

$$\Phi_i^j(d_{j'}^{\{i'\}}, p) = \begin{cases} 1 & \text{if } i = i' \text{ and } j = j' \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any $l \in \mathscr{L}(\mathcal{D}_{nm})$ with support in $\{d_j^{\{i\}} : i \in N, j \in M\}$, and any $p \in \mathscr{P}_{nm}$, we have

$$\bar{\Phi}(l, p) = \sum_{j \in M} \sum_{i \in N} l(d_j^{\{i\}}) \Phi(d_j^{\{i\}}, p) = \sum_{j \in M} \sum_{i \in N} l(d_j^{\{i\}}) e_i^j.$$

For any $\alpha \in \Delta_{nm}$, let l_α denote the random simple dictatorship in $\mathscr{L}(\mathcal{D}_{nm})$ such that $l_\alpha(d_j^{\{i\}}) = \alpha_i^j$. Since $l_\alpha = \bigoplus_{i \in N, j \in M} \alpha_i^j d_j^{\{i\}}$, we have $\bar{\Phi}(l_\alpha, p) = \alpha$ for any $p \in \mathscr{P}_{nm}$. Thus, for any $p \in \mathscr{P}_{nm}$, we have

$$\Delta_{nm} = \left\{ \bar{\Phi}(l, p) : l \in \mathscr{L}(\mathcal{D}_{nm}) \text{ s.t. } \sup(l) \subseteq \left\{ d_j^{\{i\}} : i \in N, j \in M \right\} \right\}.$$

Therefore, we have $\Delta_{nm} \subseteq \bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \{p\})$. $\square$

We now successively impose some reasonable properties on inequality indices for extended power profiles and narrow down the class of them. The first two properties bear the same implications as the axioms used by Laruelle and Valenciano (2004) to characterize their inequality index on a single issue. *Anonymity* is a reasonable property in the inequality literature and requires that an inequality measurement treat voters symmetrically or equally.

Anonymity (AN). For any extended power profile α and any permutation π of N, $I_{nm}(\alpha) = I_{nm}(\pi(\alpha))$.

The second axiom requires that an inequality measurement be rational, in the sense that the inequality index can be interpreted as a von Neumann–Morgenstern preference on at least some subdomain of extended power profiles. That is, an inequality index satisfies convex linearity or affinity on the restricted domain. Laruelle and Valenciano (2004) required this condition only for any pair of extended power profiles such that the distributions of voting power are ordinally the same. However, in fact, they require this property only for the nonincreasing distributions of voting power. We adopt the latter notion, which is relevant to the *nonincreasing comonotonic* matrices discussed by Gajdos and Waymark (2005), as found in the literature on multidimensional inequality in individuals' well-being. Formally, an extended power profile α is *nonincreasing comonotonic* if $\alpha_1^j \geq \cdots \geq \alpha_n^j$ for any $j \in M$. In addition to this modification, we require the rationality condition only on nonincreasing comonotonic extended power profiles such that the total sums of their components are identical.[8] Note that for any extended power profiles α, β in Δ_{nm}, we have $\sum_{j\in M}\sum_{i\in N}\alpha_i^j = \sum_{j\in M}\sum_{i\in N}\beta_i^j$. Then, we propose the following property.

Expected Inequality on Comonontonic Profiles (EICP). For any nonincreasing comonotonic extended power profiles α, β such that $\sum_{j\in M}\sum_{i\in N}\alpha_i^j = \sum_{j\in M}\sum_{i\in N}\beta_i^j$, and any $\lambda \in [0, 1]$,

$$I_{nm}(\lambda\alpha + (1-\lambda)\beta) = \lambda I_{nm}(\alpha) + (1-\lambda)I_{nm}(\beta).$$

The next axiom is added to the set of Laruelle and Valenciano's (2004) axioms that characterize their inequality index of power in single-issue voting situations. The axiom is concerned with the separability of an inequality index. Any issue j is considered *separable* from other issues if an inequality index is independent of the construction of subdistributions of voting powers with regards to them. That is, an inequality index for any two extended power profiles—in which only the voting power distributions on issue j are different—is independent of any voting power distributions on the remaining issues. This

[8] Laruelle and Valenciano (2004) also consider a similar restriction, to characterize their inequality measure for extended power profiles.

requirement is plausible in voting situations where an inequality index evaluates the *a priori* voting powers of various voters and also relates to a von Neumann–Morgenstern preference from a normative viewpoint.

Separability (SEP). For any $j \in M$ and any $\alpha^j, \beta^j, \bar{\alpha}^{-j}$, and $\bar{\beta}^{-j}$, $I_{nm}(\alpha^j, \bar{\alpha}^{-j}) \geq I_{nm}(\beta^j, \bar{\alpha}^{-j}) \Leftrightarrow I_{nm}(\alpha^j, \bar{\beta}^{-j}) \geq I_{nm}(\beta^j, \bar{\beta}^{-j})$.

The next result implies that the set of all nonincreasing comonotonic extended power profiles in Δ_{nm} is an $(nm-1)$-subsimplex of the simplex Δ_{nm}, whose extreme points are defined as follows:

$$\theta^{st} = (\underbrace{0, \ldots, 0}_{n(t-1) \text{ times}}, \underbrace{\tfrac{1}{s}, \ldots, \tfrac{1}{s}}_{s \text{ times}}, \underbrace{0, \ldots, 0}_{nm-n(t-1)-s \text{ times}}).$$

Lemma 1. *The set of all nonincreasing comonotonic extended power profiles* $(\alpha^1, \ldots, \alpha^m) \in \Delta_{nm}$ *is an* $(nm-1)$*-simplex whose extreme points are* $\theta^{11}, \ldots, \theta^{st}, \ldots, \theta^{nm}$. *Moreover, by setting* $\alpha^j_{n+1} = 0$ *for any* $j \in M$, *we have:*

$$(\alpha^1, \ldots, \alpha^m) = \sum_{t=1}^{m} \sum_{s=1}^{n} s(\alpha^t_s - \alpha^t_{(s+1)})\theta^{st}.$$

Proof. It is sufficient to verify that $(\alpha^1, \ldots, \alpha^m)$ can be uniquely written as the conclusion that we need. □

Theorem 1 states that any inequality index that satisfies the three aforementioned axioms can be written as follows: after obtaining a nonincreasing comonotonic extended power profile by permuting each n-dimensional vector of any extended power profile in Δ_{nm}, the nonincreasing comonotonic extended power profile is aggregated into an inequality index; then, the values of $I_{nm}(\theta^{11})$,..., $I_{nm}(\theta^{st})$,..., $I_{nm}(\theta^{nm})$ play a crucial role in determining the degree of inequality.

Theorem 1. *An index* $I_{nm} : \Delta_{nm} \to \mathbb{R}$ *satisfies AN, EICP, and SEP on* Δ_{nm} *if and only if it can be written as:*

$$I_{nm}(\alpha^1, \ldots, \alpha^m) = \sum_{t=1}^{m} \sum_{s=1}^{n} \big(sI_{nm}(\theta^{st}) - (s-1)I_{nm}(\theta^{(s-1)t})\big)\hat{\alpha}^t_s, \tag{3}$$

where $\hat{\alpha} = (\hat{\alpha}^1, \ldots, \hat{\alpha}^m)$ *is a nonincreasing comonotonic extended power profile obtained by permuting each* $\alpha^1, \ldots, \alpha^m$, *and* $I_{nm}(\theta^{0t}) = 0$ *for any* $t = 1, \ldots, m$.

Proof. It is easy to show that the index given by Equation (3) satisfies AN, EICP, and SEP. Now suppose that an index I_{nm} satisfies these axioms. By AN, we have $I_{nm}(\alpha^1, 0^{n(m-1)}) = I_{nm}(\hat{\alpha}^1, 0^{n(m-1)})$. We have $I_{nm}(\alpha^1, \alpha^2, \ldots, \alpha^m) = I_{nm}(\hat{\alpha}^1, \alpha^2, \ldots, \alpha^m)$ by SEP. Again, by AN, we have $I_{nm}(0^n, \alpha^2, 0^{n(m-2)}) = I_{nm}(0^n, \hat{\alpha}^2, 0^{n(m-2)})$. Hence, it follows from SEP that $I_{nm}(\hat{\alpha}^1, \alpha^2, \ldots, \alpha^m) = I_{nm}(\hat{\alpha}^1, \hat{\alpha}^2, \alpha^3, \ldots, \alpha^m)$. Thus, $I_{nm}(\alpha^1, \alpha^2, \ldots, \alpha^m) = I_{nm}(\hat{\alpha}^1, \hat{\alpha}^2, \alpha^3, \ldots, \alpha^m)$

can be obtained. By repeating the above arguments, we have $I_{nm}(\alpha^1, \ldots, \alpha^m) = I_{nm}(\hat{\alpha}^1, \ldots, \hat{\alpha}^m)$. By Lemma 1 and EICP, we obtain:

$$\begin{aligned} I_{nm}(\hat{\alpha}^1, \ldots, \hat{\alpha}^m) &= I_{nm}\left(\sum_{t=1}^{m}\sum_{s=1}^{n} s(\hat{\alpha}^t_s - \hat{\alpha}^t_{(s+1)})\theta^{st}\right) && \text{(by Lemma 1)} \\ &= \sum_{t=1}^{m}\sum_{s=1}^{n} s(\hat{\alpha}^t_s - \hat{\alpha}^t_{(s+1)}) I_{nm}(\theta^{st}) && \text{(by EICP)} \\ &= \sum_{t=1}^{m}\sum_{s=1}^{n} \big(s I_{nm}(\theta^{st}) - (s-1) I_{nm}(\theta^{(s-1)t})\big)\hat{\alpha}^t_s. \end{aligned}$$

□

In the single-issue case, Laruelle and Valenciano (2004) impose further axioms to narrow down the class of the inequality indices and single out one index. Before we formalize their axioms in our model, we introduce some definitions. For any nonempty subset $U \subseteq N$ and any issue $j \in M$, let θ^{Uj} be the extended power profile such that U-components for issue j are $\frac{1}{u}$ and any other components of the profile are 0, where u is the cardinality of U. The profile θ^{Uj} captures the situation where only issue j is a nontrivial collective decision problem and each voter in U has equal voting power with regards to j. In such a situation, Laruelle and Valenciano (2004) impose the following two properties on inequality indices. The first requires that an inequality index be constantly sensitive to the addition of null-voters. The second requires that the degree of inequality be minimal when all voters have equal voting power; then, its value is normalized to zero. We require these properties for any issue $j \in M$.

Constant Sensitivity to Null-voters (CSN). For any $U, V \subseteq N$, any $i \in N\backslash U$, any $i' \in N\backslash V$, and any $j \in M$, $I_{nm}(\theta^{Uj}) - I_{nm}(\theta^{U\cup\{i\}j}) = I_{nm}(\theta^{Vj}) - I_{nm}(\theta^{V\cup\{i'\}j}) > 0$.

Zero Normalization (ZN). For any $j \in M$, $I_{nm}(\theta^{Nj}) = 0$.

Lemma 2 shows the logical relationship of the above axioms, and AN is implied by the joint satisfaction of EICP and CSN.

Lemma 2. *If an index $I_{nm} : \Delta_{nm} \to \mathbb{R}$ satisfies EICP and CSN, then it satisfies AN.*

Proof. Let I_{nm} be an index satisfying EICP and CSN. For any $j \in M$, define $S^j := I_{nm}(\theta^{Uj}) - I_{nm}(\theta^{U\cup\{i\}j})$, which by CSN does not depend on $U \subseteq N$ and $i \in N$. Let $u < n$. By applying CSN $(n-u)$ times, we can easily obtain $I_{nm}(\theta^{Uj}) = I_{nm}(\theta^{Nj}) + (n-u)S^j$, which implies that $I_{nm}(\theta^{Uj})$ depends only on u. Therefore, if $u = v$, then we have $I_{nm}(\theta^{Uj}) = I_{nm}(\theta^{Vj})$, where v is the cardinality of V. Now take any $\alpha \in \Delta_{nm}$. By Lemma 1, α can be uniquely written as a convex combination of the extreme points of an $(nm-1)$-simplex of comonotonic extended power profiles. If these extreme points are

$\theta^{U_1 j}, \theta^{U_2 j}, \ldots, \theta^{U_n j}$ for any $j \in M$, where the cardinality of U_s is s, we have $\alpha = \sum_{t=1}^m \sum_{s=1}^n \lambda_s^t I_{nm}(\theta^{U_s t})$ (for some $\lambda_s^t \geq 0$ such that $\sum_{t=1}^m \sum_{s=1}^n \lambda_s^t = 1$). Then, by EICP, we have $I_{nm}(\alpha) = \sum_{t=1}^m \sum_{s=1}^n \lambda_s^t I_{nm}(\theta^{U_s t})$. However, we then have $I_{nm}(\alpha) = I_{nm}(\pi(\alpha))$ for any permutation π of N, because each $I_{nm}(\theta^{U_s t})$ depends only on s and t, which implies that I_{nm} satisfies AN. $\square$

Theorem 2 shows that the remaining four axioms characterize (up to positive proportionality constants) an inequality index on Δ_{nm}, which is a two-stage aggregation procedure where the first-stage aggregator is based on the Laruelle–Valenciano measure for each issue and the second-stage aggregator sums up the inequality index on each issue.

Theorem 2. *There exists a unique (up to positive proportionality constants S^t for any $t = 1, \ldots, m$) inequality index $I_{nm} : \Delta_{nm} \to \mathbb{R}$, which satisfies EICP, SEP, CSN, and ZN, and it is given by*

$$I_{nm}(\alpha^1, \ldots, \alpha^m) = \sum_{t=1}^m \big(S^t \sum_{s=1}^n (n - 2s + 1)\hat{\alpha}_s^t\big). \tag{4}$$

Proof. First, it is easy to show that the index given by Equation (4) satisfies the axioms in Theorem 2. Now, suppose that an index I_{nm} satisfies them. By Lemma 2, it also satisfies AN. Therefore, by Theorem 1, $I_{nm}(\alpha^1, \ldots, \alpha^m)$ is given by Equation (3). We define $S^j := I_{nm}(\theta^{Uj}) - I_{nm}(\theta^{U \cup \{i\} j})$ for any $j \in M$, where S^j does not depend on any pair i, U such that $i \notin U \subseteq N$ because I_{nm} satisfies CSN. It immediately follows from ZN that $I_{nm}(\theta^{st}) = (n - s)S^t$ for $s = 1, \ldots, n$ and $t = 1, \ldots, m$. Then, by substituting them in Equation (3), we can obtain Equation (4). $\square$

In the next step, given this index on the subdomain Δ_{nm} of extended power profiles, we extend it to the whole domain $\bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm})$, and any restricted domain $\bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \{p\})$ for any $p \in \mathscr{P}_{nm}$. As in the Laruelle and Valenciano (2004) model, any extended power profile in those domains is proportional to an extended power profile in Δ_{nm}, which by Proposition 1 is contained in all those domains. Therefore, by imposing the four axioms in Theorem 2 on Δ_{nm}, we obtain a *relative index* of inequality that ranks all extended power profiles in the whole domain. An inequality index is relative if $I_{nm}(\alpha) = I_{nm}(k\alpha)$ for any extended power profile α and any $k > 0$.

Therefore, there is a unique (up to positive proportionality constants S^t for any $t = 1, \ldots, m$) relative inequality index on $\bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm})$, and on $\bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \{p\})$ for any $p \in \mathscr{P}_{nm}$, that satisfies EICP, SEP, CSN, and ZN on Δ_{nm}, and it is given by

$$I_{nm}(\alpha^1, \ldots, \alpha^m) = \sum_{t=1}^m \Big(S^t \sum_{s=1}^n (n - 2s + 1) \frac{\hat{\alpha}_s^t}{\sum_{j \in M} \sum_{i \in N} \alpha_i^j}\Big). \tag{5}$$

Equation (5) represents a two-stage aggregation procedure where, in the first stage, the distribution of voting power with regards to each issue is aggregated into an inequality index based on the Laruelle–Valenciano measure. Then, the second-stage aggregator sums up the inequality index on each issue. As Laruelle and Valenciano (2004) mention, if we choose $S^t = \frac{1}{n}$ for $m = 1$ as the value of the constant, the Laruelle–Valenciano measure is equivalent to the Gini index discussed in the literature on inequality of income distributions.

Consider the case where the four axioms in Theorem 2 are imposed on the whole domain. Laruelle and Valenciano (2004) first required their EICP axiom for *any* pair of extended power profiles such that the distributions of voting power are ordinally the same. However, they had to use a restricted version of their EICP axiom, as defined in this paper, to characterize their inequality index on the whole domain. This is because, as Laruelle and Valenciano (2004) point out, an unrestricted EICP axiom joined with CSN and ZN implies an absolute index—something that is incompatible with relative indices. Because the same problem is also encountered with our model, we define EICP in the restricted manner.

Finally, we summarize the above discussions in Theorem 3.

Theorem 3. *There exists a unique (up to positive proportionality constants S^t for $t = 1, \ldots, m$) relative inequality index $I_{nm} : \bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm}) \to \mathbb{R}$ that satisfies EICP, SEP, CSN, and ZN; it is given by Equation (5).*

By Proposition 1, this characterization is valid when we restrict the domain of an inequality index to the form $\bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \{p\})$ for any $p \in \mathscr{P}_{nm}$.

6 Variable Numbers of Voters and Issues

In this section, we consider the case where both the number of voters and the number of issues are variable. In many voting situations, these two numbers are not fixed. For example, the number of bills submitted each year and the number of legislators who vote on them are variable. Thus, in general, we can define an inequality index of voting powers as $I : \bigcup_m \bigcup_n \bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm}) \to \mathbb{R}$. On the other hand, the axioms discussed in Section 5 are imposed on the restriction of such a whole index to extended power profiles with any number of voters and any number of issues. Then, we could obtain a family of indices $I = \{I_{nm} : \bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm}) : n = 2, 3, \ldots, \text{ and } m = 1, 2, \ldots\}$ with a family of positive constants $\left(S^t_{nm}; t = 1, \ldots, m\right)_{n,m \in \mathbb{N}, n \neq 1}$.

We now work to characterize a specific index I by assuming equivalence principles relative to the number of voters and the number of issues. In a single-issue case, Laruelle and Valenciano (2004) propose two equivalence principles. The first principle requires that a dictatorship has a common degree of inequality, irrespective of the number of voters. The second principle requires that if the number of null-voters is the same, the degree of inequality remains the same—inasmuch as voting power is equally shared by the remaining voters—regardless of the number of voters. We extend these two principles to the case of multiple issues.

Simple Dictatorship Equivalence Principle (SDEP). For any $n, m \in \mathbb{N}$, any $j \in M$, and any α, β such that any issue $j' \neq j$ is trivial and

$$\alpha^j = (1, \underbrace{0, \ldots, 0}_{n \text{ times}}) \text{ and } \beta^j = (1, \underbrace{0, \ldots, 0}_{n+1 \text{ times}}),$$

$I(\alpha) = I(\beta)$.

Simple Null-voter Equivalence Principle (SNEP). For any $n, m \in \mathbb{N}$, any $j \in M$, and any α, β such that any issue $j' \neq j$ is trivial and

$$\alpha^j = (\underbrace{\tfrac{1}{s}, \ldots, \tfrac{1}{s}}_{s \text{ times}}, \underbrace{0, \ldots, 0}_{n-s \text{ times}}) \text{ and } \beta^j = (1, \underbrace{0, \ldots, 0}_{n-s \text{ times}}),$$

$I(\alpha) = I(\beta)$.

By using either of the two equivalence principles, we can narrow down a family of positive constants $\left(S^t_{nm}; t = 1, \ldots, m\right)_{n,m \in \mathbb{N}, n \neq 1}$. For any $m = 1, 2, \ldots,$ SDEP assigns a common maximum of inequality to each simple dictatorship for any number of voters. Therefore, given the number of issues m, we have $S^t_{nm} = (\frac{1}{n-1})S^t_m$ for $t = 1, \ldots, m$, where S^t_m is an arbitrary positive constant. On the other hand, given the number of issues m, SNEP implies that all of the positive constants are equal for any $t = 1, \ldots, m$—that is, $S^t_{nm} = S^t_m$ for $t = 1, \ldots, m$, where S^t_m is an arbitrary positive constant.

Next, we consider an equivalence principle when the number of issues is variable. Compare the following two decision rules. One is a dictatorship on a single issue, and the other is a decision rule under which voting power with regards to each issue is almost equally shared by all voters. Under the latter decision rule, the Laruelle–Valenciano index on each issue is slightly positive. Then, if an inequality index is simply the total sum of the Laruelle–Valenciano indices, the overall index in the latter could be greater than that of the former when the number of issues in the latter is sufficiently large; this is counterintuitive. With this problem in mind, we set up the maximal inequality for extended power profiles when the number of issues varies.

A reasonable starting point is to assign a common maximal degree of inequality to decision rules in which there exists a common dictator on each issue, regardless of the number of issues. Formally, we define this principle as follows.

Dictatorship Normalization (DN). For any $n, m, m' \in \mathbb{N}$ and any α, β such that any $\alpha^j = (1, \ldots, 0)$ for any $j \in M$ and $\beta^{j'} = (1, \ldots, 0)$ for any $j' \in M'$,

$$I(\alpha) = I(\beta),$$

where m and m' are the cardinality of M and M', respectively.

Thus, any extended power profiles in which there exists a common dictator on each issue correspond to the maximal inequality, regardless of the number of issues. This implies that $S^t_m = \frac{1}{m}S$, where S is an arbitrary positive constant. Therefore, we have two indices, depending on whether SDEP or SNEP is assumed, up to a positive constant. We summarize the above observations in Theorem 4.

Theorem 4. *There exists a unique (up to a positive proportionality constant) inequality index* $I : \bigcup_m \bigcup_n \bar{\Phi}(\mathscr{L}(\mathcal{D}_{nm}) \times \mathscr{P}_{nm}) \to \mathbb{R}$ *that satisfies EICP, SEP, CSN, and ZN for any* $n, m \in \mathbb{N}$, $n \neq 1$, *and that satisfies DN, and either SDEP or SNEP. They are respectively given, up to a positive proportionality constant, by:*

$$I^{DP}(\alpha^1, \dots, \alpha^m) = \frac{1}{m(n-1)} \sum_{t=1}^{m} \Big(\sum_{s=1}^{n} (n - 2s + 1) \frac{\hat{\alpha}^t_s}{\sum_{j \in M} \sum_{i \in N} \alpha^j_i} \Big), \quad (6)$$

$$I^{NP}(\alpha^1, \dots, \alpha^m) = \frac{1}{m} \sum_{t=1}^{m} \Big(\sum_{s=1}^{n} (n - 2s + 1) \frac{\hat{\alpha}^t_s}{\sum_{j \in M} \sum_{i \in N} \alpha^j_i} \Big). \quad (7)$$

From Equations (6) and (7), it is possible to interpret that our two inequality indices of powers in multiple-issue voting situations are the arithmetic means of the two Laruelle–Valenciano indices on each issue. As in Laruelle and Valenciano (2004), the two indices differ only in a multiplicative factor that depends only on the number of voters and the number of issues, and which have the following relationship:

$$I^{DP} = \frac{I^{NP}}{n-1}.$$

We now make a few remarks on our inequality indices of voting powers. Although those indices are simple and reasonable from a normative viewpoint, they might create problems in real-world voting situations. In the example offered in the Introduction of decisions made by the UN Security Council, nonprocedural matters may not be considered as important as procedural matters. Since our inequality indices impose the same weight on each issue, it is impossible to capture the varying degrees of importance among the various issues. Thus, it might be restrictive for a positive proportionality constant S^t_m to be the same for any $t = 1, \dots, m$, because it could represent the degree of importance of an issue. Thus, the problem of assigning a degree of importance to each issue is left unresolved, in those cases where one would like to measure the inequality of voting powers in practical voting situations.

7 Concluding Remarks

In this study, we extend the inequality measure of voting power proposed by Laruelle and Valenciano (2004) to a case in which there are multiple issues on which to vote. Instead of simple games, our model is based on ternary voting games; for each issue, each voter has three voting options—namely, “yes,” “no,” and “abstention.” In voting situations, voting power is defined as the probability of there being voting profiles in which a voter will play a crucial role in collective decision-making. We propose a two-stage aggregation procedure where in the first stage, the distribution of voting power with regards to each issue is aggregated into an inequality index based on the Laruelle–Valenciano measure; then, the second-stage aggregator sums up the inequality index on each issue.

This study provides an axiomatic foundation of our inequality measure of voting powers by adding an axiom, Separability (SEP), to the set of axioms that characterize the Laruelle–Valenciano inequality measure of voting power. In the case where the number of voters and the number of issues are variable, we propose three equivalence principles with respect to the number of voters or the number of issues, two of which are similar to those of Laruelle and Valenciano (2004). Then, we show that our inequality measure of voting powers can be expressed as the arithmetic mean of the Laruelle–Valenciano inequality index of voting power with regards to each issue.

Finally, we highlight an open question otherwise not addressed by this study. Suppose that 2 voters and 2 issues exist. Consider the following two decision rules—that is, voter 1's dictatorship on both issues, and the combination of voter 1's dictatorship on issue 1 and voter 2's dictatorship on issue 2. Our inequality measure of voting powers assigns the same maximal inequality to the two decision rules. However, the former might be more unequal than the latter, because while the former is uneven, the latter is even in terms of the share of issues on which voters have decisive power, even if both decision rules have a dictator on each issue.[9]

In fact, from the proof of Theorem 1, it is easy to see that the same problem is valid for any inequality index that satisfies both SEP and Anonymity (AN). AN and SEP are both reasonable if an inequality index evaluates the *a priori* voting powers of various voters and relates to a von Neumann–Morgenstern preference from a normative viewpoint. Therefore, one possible approach in addressing the problem is to apply nonexpected utility theory under uncertainty (e.g., Gilboa and Schmeidler 1989) to the measurement of inequality of voting powers; however, we leave this problem to future research.[10]

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[9] This problem has a structure similar to that of "positive dependency" between attributes that characterize individuals' well-being. That is, although inequality of voting powers can be reduced only by deducing positive dependence between the row vectors of extended power profiles, our inequality index does not satisfy this property. See Gajdos and Waymark (2005) and Tsui (1999).

[10] See also Ben-Porath et al. (1997) and Gajdos and Maurin (2004) for the measurement of inequality of income distributions under uncertainty.

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