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Reflections on Hypernite Approaches to
General Equilibrium Analysis and Welfare
Economics: “Back to the Basics”

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Reflections on Hyperfinite Approaches to General Equilibrium Analysis and Welfare Economics: "Back to the Basics"*†

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Abstract

This article compiles my recent and on-going investigations of the existence of general equilibria and their welfare assessments in "large-square" economies, with a special emphasis on demonstrations of the strength of the hyperfinite methodology over the alternative measure-theoretic approach.

Hyperfinite analysis has emerged as a powerful alternative in the proofs where finite/infinite linkages are crucial and subtle, as evident in AVERAGE CONVEXITY THEOREMS and EXTENSION PROPERTIES, on the one hand, and derivations of finite implications in the forms

*This is an outgrowth from my Final Lecture delivered on February 4, 2015 on the occasion of my retirement from Chiba University, and therefore mainly retrospective in nature. However, newly supplemented SECTION **5** gives an extensive account of the strands of my on-going researches at Nishogakusha University to be reported in my Monograph [3] (In Preparation).

†**Disclaimer:** Throughout this article, three nomenclatures "Hyperfinite Analysis," "Hyperreal Analysis" and "Nonstandard Analysis" are used interchangeably, as many researchers in this field tend to do with particular "superstructures" on mind of Finite Numbers, Real Numbers, and Standard Analysis, respectively, depending on the nature of the problems we tackle.

‡**Acknowledgments:** I thank my former and present colleagues for long-time pleasant association and continued moral support.

of ASYMPTOTIC INTERPRETATIONS and ELEMENTARY THEOREMS, on the other. In quite a few interesting cases we have managed to get rid of "external" arguments from the original hyperfinite proofs, to complete elementary proofs, explicating the convergence speeds in terms of the size of the economy.

The Loeb measure has also proven complementary, rather than alternative, in establishing standard measure-theoretic results for a large class of measure spaces that arise in economics, by converting the comparable results established first by hyperfinite analysis.

Besides the well cultivated research subjects, the present report includes, as promising applications, some important but overlooked research agenda concerning the mixed markets with "large" agents together with an ocean of "small" agents, that may help to shed light on the exact nature of the market power in the presence of an infinite variety of differentiated commodities.

Contents

1 Introduction

1.1 The Scope of Researches

In order to investigate the existence of general equilibria and their welfare assessment comprehensively in such diverse economic situations as intertemporal resource allocations over an infinite time horizon, incomplete markets under uncertainty, commodity differentiation, and economic location, I have idealized perfectly competitive economies arising in the afore mentioned situations by "Large-Square (or Large^2) Economies," i.e., with infinitely many agents and infinitely many contingent commodities, with a due emphasis on the appropriate choice of "myopic" topologies endowed on the commodity space in the respective context.

- **Economic Myopia:** More specifically, implicit commonly in continuous preferences in such "myopic" topologies as the *Mackey topology* $\mathcal{T}_M$ in the context of intertemporal exchanges and the *weak-star topology* $\mathcal{T}_{w^*}$ for commodity differentiation is the myopic economic behavior in the sense that considerations of sufficiently large but finitely many matters dominate the agent's preference judgments with the additional considerations of matters in the "tail" being incapable of overturning the original judgment[1]. This will suffice to ensure the EXTENSION PROPERTY to be exploited in THEOREM 2 of SECTION **2** in order to extend the sufficiently large number of partial equilibrium analyses to a full fledged general equilibrium analysis.

- **Average Convexity:** For a general class of "Large-Square Economies" which admit nonconvex preferences typical of risk (or extremity) loving preferences and/or the nonconvex commodity space due to commodity indivisibility in the context of exchange economies (plus nonconvex technologies exhibiting increasing returns to scale, when the economies are extended to include production activities), I have established the emergence of pseudo-markets by way of the existence of approximately competitive equilibria therein, and their welfare properties as stated in terms of the equivalence of approximate equilibrium allocations and their cooperative counterparts, in particular, the core, the value and the bargaining set allocations.
  - Even in the presence of infinitely many commodities, individual anomalies due to nonconvex preferences are "approximately" salvaged as average convexity as well as upper hemicontinuity of the aggregate demand, when summed over "sufficiently many" agents.
  - Even with an introduction of infinitely many indivisible commodities, a postulate for (at least) one perfectly divisible commodity, say "money" together with accordingly modified preferences, will suffice to contain those making anomalous choices within a "negligible" subgroups, and consequently restore the desired upper hemicontinuity of aggregate demand, thus the money is functioning to smoothen out transactions.

It turns out that incorporations of nonconvexities have particularly awakened our awareness of the underlying similarity of the "Market Thickness" Requirement and the"Relative Size" Requirement between the space of agents and that of commodities. This will be discussed in **5.3.4**.

Besides the perfectly competitive extremity, I became aware that important overlooked research agenda await serious investigations, concerning the

[1]This is quite in contrast to the continuous preferences on *finite dimensional* commodity spaces, where all topologies but the *discrete topology* are equivalent in the sense that continuous preferences in one topology are also continuous in other topologies. Therefore, "continuity" of preferences on finite dimensional commodity spaces is not a behavioral hypothesis, but a mere mathematical requirement, or at best in exclusion of such anomalies as discontinuous choices inherent to "Lexicographic orderings."

mixed markets with "large" agents together with an ocean of "small" agents, that may help to shed light on the exact nature of the market power in the presence of an infinite variety of differentiated commodities. Some findings are reported in **5.3.6**.

1.2 The Methodology

I have applied Nonstandard (Hyperreal, or Hyperfinite) Analysis to General Equilibrium Analysis and Welfare Economics (Nomura (1981), (1984a), (1984b), (1985), (1989), (1986), (1992c), (1993a), (1993b), (1995), (1998a) and (1999)).

I have demonstrated convincingly the strength of Nonstandard Analysis over the alternative "continuum" characterization in deducing asymptotic interpretations and preferably elementary proofs for the "real" finite economies of the limit results on the existence of general equilibria and their welfare assessments established for the limit *double-hyperfinite* "Large-Square Economies."

As two such strengths, I may emphasize on

(1) the convergence speed explicated in Elementary Theorems (Nomura (1984a), (1985), (1986), (1992c), (1993a), (1995) and (1998a)) via Nonstandard Analysis in terms of the ever increasing number of agents,

and on

(2) the "market thickness," or the "relative size" requirement that the number of commodities be "sufficiently smaller" than the number of agents, in the specific sense that may be deduced from the hypothesis explicated in the Nonstandard analogue of Shapley-Folkman Average Convexity Theorem (Nomura (1981), (1992c), (1993a) and (1995)).

The strength (1) is all the more evident in the analysis of mixed markets with "large" traders which seem to have been overlooked for a long time, and invincibility of which may be overturned with the aid of hyperfinite analysis. We shall take up this problem and report some new results in the subsequent **5.3.6**

1.3 Citations

- My Ph.D. dissertation at the Johns Hopkins University, NOMURA (1984a), supervised by Professors M. Ali Khan and Peter K. Newman, consists of Three Essays: ESSAY 1 was circulated as NOMURA (1984b); ESSAY 2 was presented to the North American Winter Meeting of the Econometric Society, Washington D.C. as NOMURA (1981); and ESSAY 3, the revision of which was later published in *Journal of Economic Theory* as NOMURA (1993a). This dissertation is cited in Salim RASHID's monograph (1987) on the applications of Nonstandard Analysis to the studies of large economies.

- My *Journal of Economic Theory* paper, NOMURA (1993a), is recognized by PierCarlo NICOLA (2000, p.405) among *Mainstream Mathematical Economics in the 20th Century* (Title of the Book) as: "Nomura (1993[a]) studies the existence of approximate equilibria [of infinite dimensional economies] under nonconvex preferences."

- The usefulness of an extension of SHAPLEY-FOLKMAN AVERAGE CONVEXITY THEOREM to infinite dimensional ranges established in the above NOMURA (1993a, LEMMA 1; Also quoted as THOEREM 8 in the subsequent SECTION **4**) is not tarnished even today, as vindicated by the fact that it is still quoted in the recent *Positivity* paper by Ali KHAN and Kali RATH (2013).

- By a resort to the preceding SHAPLEY-FOLKMAN AVERAGE CONVEXITY THEOREM OF INFINITE DIMENSIONAL RANGES, I circulated CORE EQUIVALENCE THEOREMS FOR LARGE-SQUARE ECONOMIES in NOMURA (1992c, THEOREM 1; Quoted as THEOREM 6 in the subsequent SECTION **3**), and presented BARGAINING SET EQUIVALENCE THEOREMS FOR LARGE-SQUARE ECONOMIES to the Econometric Society Seventh World Congress, and is circulated as NOMURA (1995, THEOREM 1; Quoted as THEOREM 7 in SECTION **3**). NOMURA (1992c) received citations from Bob ANDERSON and Bill ZAME's *Econometrica* paper (1997), and their *Economic Theory* paper (1998) with detailed discussions.

2 Existence of Approximate Equilibria in Large-Square Exchange Economies

$\mathbf{R}_\infty(\mathbf{N})$, or $\mathbf{R}_\infty$ in short, is the space of real-valued sequences. Its subspaces $\mathbf{R}^+_\infty$, $\mathbf{R}^d_\infty$, to be introduced in the subsequent Theorems, or the bounded subspaces ℓ^+_∞, ℓ^d_∞ with the norm $\|\ \|_\infty$ will serve as the commodity spaces over an infinite horizon, or the space of Mas-Colell's "individual commodity bundles" defined on the space of commodity characteristics (Nomura (1981, 1992c, 1993a, and 1995)).

The commodity spaces will be endowed alternatively with the Mackey topology $\mathcal{T}_M$ or the weak topology $\mathcal{T}_w$, which is the strongest or the weakest topology, respectively, that is consistent with the duality $\langle \ell_\infty, \ell_1 \rangle$.

Alternatively, a Hilbert space L_2 of square-integrable martingales is a natural candidate for the space of state-contingent claims under uncertainty, with arbitrage prices taking values in its dual L_2 (Nomura (1986)).

The space of preferences $\mathcal{P}_{mo}$, $\mathcal{P}_{odd}$, $\mathcal{P}_{idd} \subset \mathcal{P}((\mathbf{R}^+_\infty, \mathcal{T}) \times (\mathbf{R}^+_\infty, \mathcal{T}))$, which need *not* be convex, will be endowed with Narens' compact topology $\mathcal{C}$ on closed subsets or the Hausdorff uniformity $\mathcal{U}_H$ induced from $\mathcal{T}_M$ or $\mathcal{T}_w$ on ℓ_∞.

Transfer Principle: *By transfer*, with respect to *both* n and k, of the standard existence result of approximate equilibria for a finite exchange economy $\mathcal{E}_{n,k} : A \to \mathcal{P}_{mo} \times \mathbf{R}^k_+$ with $|A| = n \in \mathbf{N}$ and $k \in \mathbf{N}$, and after some fine tuning, we have:

Theorem 1 (Nomura (1981, Theorem 3), Nonstandard Truncated). *Let an internal map $\mathcal{E}_{\nu,\omega} : A \to {}^*\mathcal{P}^\nu_{mo} \times {}^*\mathbf{R}^\nu_+$ with $|A| = \omega \in {}^*\mathbf{N} - \mathbf{N}$ be a double hyperfinite exchange economy with the ν-dimensional truncated commodity space, where $\nu \in {}^*\mathbf{N} - \mathbf{N}$.*

Suppose $\mathcal{E}_{\nu,\omega}$ satisfies

(a) *Integrability of Initial Endowments: For all $a \in A$, $\|e(a)\|_1$ is finite. $\dfrac{\sum_{a\in A} e(a)}{\omega} \gtrapprox 0$, and $e(a)$ is $\mathcal{T}_M$- or $\mathcal{T}_w$-S-integrable,*

(b) *Compactness of Preferences: For all $a \in A$, $\succ_a \in {}^*\mathcal{P}^\nu_{mo}$ is $\mathcal{C}$-near-standard,*

and

(c) *Market Thickness*: $\dfrac{\nu}{\sqrt{\omega}} \simeq 0$.

Then, there exist $p \in \left\{ p \in {}^*\mathbf{R}^{\nu}_{+} \middle| \, \|p\|_{\infty} \leq 1,\, p^i \geq \dfrac{1}{\sqrt{\omega}} \ (\forall i \in \{1, ..., \nu\}) \right\}$, *an assignment* $g : A \to {}^*\mathbf{R}^{\nu}_{+}$, *and an internal subset* $S \subset A$ *such that*

(1) $\dfrac{|S|}{\omega} \simeq 1$,

(2) *for all* $a \in S$, $g(a) \in d^{\nu}(p, a)$,

and

(3) $\dfrac{\sum_{a \in A} g(a)}{\omega} \lesssim 0$.

Moreover,

(4) p *is* $\mathcal{T}_M$*- or* $\mathcal{T}_w$*-near-standard, and* $p^i \gtrsim 0$ *for all* $i \in \{1, ..., \nu\}$,

and

(5) *there exists* $T \subset S$ *such that* $\dfrac{|T|}{\omega} \simeq 1$, *and for all* $a \in T$, $g(a) + e(a)$ *is* $\mathcal{T}_M$*- or* $\mathcal{T}_w$*-near standard.*

Extension Property: THEOREM 2 establishes the extensibility of the *partial* equilibrium existence result (THEOREM 1) with ${}^*\mathbf{R}^{\nu}_{+}$, a hyperfinite ν-truncation of the listing of commodities, to the *general* equilibrium result with the full-fledged *-transform ${}^*\mathbf{R}^{+}_{\infty}({}^*\mathbf{N})$ of the basic commodity space $\mathbf{R}^{+}_{\infty}(\mathbf{N})$. This step takes care of the *external* nature of the countable infinity by resorting to the *economic myopia* implicit in the $\mathcal{T}_M$- or $\mathcal{T}_w$-continuous preferences, i.e., $(\forall x, y, z \in \ell^{+}_{\infty} \subset \mathbf{R}^{+}_{\infty})\ x \succ y \Rightarrow x \succ y + \hat{z}_k$ for sufficiently large $k \in \mathbf{N}$, where $\hat{z}_k$ is the "tail" of z consisting of $\hat{z}^i_k = 0$ for $i = 1, ..., k$ and $\hat{z}^i_k = z^i_k$ for $i = k + 1, ...$

THEOREM 2 (NOMURA (1981, THEOREM 5), Nonstandard). *Let* $\mathcal{E}_{\omega} : A \to {}^*\mathcal{P}_{mo} \times {}^*\mathbf{R}^{+}_{\infty}$ *be a hyperfinite exchange economy constructed from* $\mathcal{E}_{\nu,\omega}$ *by choosing.*

(i) *the identical set of traders* A, *where* $|A| = \omega \in {}^*\mathbf{N} - \mathbf{N}$,

and by assigning to each $a \in A$

(ii) $\succ'_a$, *the nonstandard extension of* $^\circ \succ_a$, *the standard part of* $\succ_a \in {}^*\mathcal{P}_{mo}$, *and*

(iii) $I(a)$, *the nonstandard extension of* $^\circ e(a)$, *the standard part of* $e(a)$.

Let $p \in \left\{ p \in {}^*\mathbf{R}^+_\infty \middle| \ \|p\|_\infty \leq 1,\ p^i \geq \dfrac{1}{\sqrt{\omega}}\ (\forall i \in {}^*\mathbf{N}) \right\}$, $g : A \to {}^*\mathbf{R}^+_\infty$ *and* $S \subset A$ *be as described in* Theorem 1. *Denote by* p_c *the nonstandard extension of* $^\circ p$, *the standard part of* p, *and write* $p = p_c + p_\infty$.

Define $f : A \to {}^*\mathbf{R}^+_\infty$ *by*

$$f^i(a) = \begin{cases} g^i(a) + e^i(a) - I^i(a) - \frac{p^i_\infty(I^i(a) - e^i(a) - g^i(a)))}{p^i_c} & \text{for } 1 \leq i \leq \nu \\ \frac{p.(I(a) - e(a))}{p_c^{\nu+1}} & \text{for } i = \nu + 1 \\ 0 & \text{for } i \geq \nu + 2. \end{cases}$$

Then,

(1) $p_c \in {}^*\ell^+_1$,

(2) *for all* $a \in S$, $f(a) \in d(p_c, a)$,

and

(3) $\dfrac{\sum_{a \in A} f(a)}{\omega} \lesssim 0$.

Asymptotic Interpretations[2]**:** The *asymptotic interpretation* (Theorem 3) for a properly defined sequence of finite economies $\{\mathcal{E}_n\}$ follows by contradiction: Suppose to the contrary. Then, we can find a contradiction to what we have proven in Theorem 2 for the nonstandard limit economy $\mathcal{E}_\omega$.

Construct from $\mathcal{E}_\omega$ a sequence of finite exchange economies $\{\mathcal{E}_n\}$ by the following procedure:

(i) Choose a finite subset $A_n \subset A$ so that $|A_n| \to \infty$ as $n \to \infty$;

(ii) Assign to each $a \in A$, $^\circ \succ_a$ and $^\circ e(a)$, the standard parts of $\succ_a$ and $e(a)$, respectively.

[2]More detailed discussions will be given in the subsequent **5.3.1**.

THEOREM 3 (NOMURA (1981, THEOREM 1), Asymptotic Interpretation). *Let* $\mathcal{E}_n : A_n \to \mathcal{P}_{mo} \times \mathbf{R}^+_\infty$ *be a sequence of finite exchange economies as defined above.*

Then,

(1) *there exists* $K \subset \mathcal{P}_{mo}$ *compact in* $\mathcal{C}$ *such that* $^\circ \succ_a \in K$ *for all* $a \in A_n$*, and*

(2) $\dfrac{\sum_{a\in A_n} {}^\circ e(a)}{|A_n|} < \infty$*, and* $E_n \subset A_n$ *and* $\dfrac{|E_n|}{|A_n|} \to 0$ *imply* $\dfrac{\sum_{a\in A_n} {}^\circ e(a)}{|A_n|} \to 0$.

Moreover, for any $\delta > 0$*, there exists* $\bar{n} \in \mathbf{N}$ *such that, for every* $\mathcal{E}_n$*,* $n \geq \bar{n}$*, there exist a price* $p_n \in \ell_1^+$ *and a net assignment* $f_n : A_n \to \mathbf{R}_\infty$ *satisfying*

(3) $\dfrac{|\{a \in A_n |\ f_n(a) \in d_n(p_n, a)\}|}{|A_n|} \geq 1 - \delta,$

and

(4) $\dfrac{\sum_{a\in A_n} f_n(a)}{|A_n|} \leq \delta.$

Elementary Approach[3]**:** It is high time to exploit the insights gained in carrying out the nonstandard proof of THEOREMS 1 and 2, and build the following *elementary* proof for the fixed finite economy $\mathcal{E}$ from the scratch.

Assumption 1 (Finite Spannability): Every $d(p, a)$ has a finite family of convex subsets $d^j(p, a)$, $j = 1, ..., \kappa(a)$ of $d(p, a)$ such that

$$d(p, a) = \bigcup_{j=1}^{\kappa(a)} d^j(p, a).$$

For $d(p, a) \subset \mathbf{R}_\infty$, the *radius* of $d(p, a)$ is $\mathrm{rad}(d(p, a)) = \inf_{x\in\mathbf{R}_\infty} \sup_{y\in d(p,a)} \|x - y\|_2$, and the *inner radius* of $d(p, a)$ is $\mathrm{r}(d(p, a)) = \sup_{x\in \mathrm{con}\, d(p,a)} \inf_{S\subset d(p,a) \text{ spans } x} \mathrm{rad}(S)$.

[3]More detailed discussions on the scope and the limitations of the Elementary Approach will be given in the subsequent **5.3.2**.

Assumption 2 (Bounded Nonconxexity): There exits an $M \in \mathbf{R}^{+}$, such that $\mathrm{r}(d(p,a)) \leq M$ for all $a \in A$.

THEOREM 4 (Nomura (1993a, THEOREM 1), Elementary). *Let* $\mathcal{E} : A \to \mathcal{P}_{mo} \times \mathbf{R}^{+}_{\infty}$ *with* $|A| = n$ *be a finite exchange economy with countably many commodities. Then, there exist* $p \in \Delta = \left\{ p \in \mathbf{R}^{+}_{\infty} \middle| \; p. \sum_{a \in A} e(a) \leq 1, p \gg 0 \right\}$ *and a net allocation* $g(a) \in \mathsf{con}\, d(p,a)$.

Suppose further that $\mathcal{E}$ *satisfies Assumptions* 1 *and* 2.

Then, for any such g*, there exists a selection* $f(a) \in d(p,a)$ *such that*

$$\frac{1}{n} \left\| \sum_{a \in A} f(a) \right\|_2 \leq \frac{1}{n} \left\| \sum_{a \in A} (f(a) - g(a)) \right\|_2 \leq \frac{2M}{\sqrt{n}}.$$

For a *type-decomposable* economy to be defined in the subsequent Corollary 1, Assumptions 1 and 2 take the following forms, respectively.

Assumption 1' (Finite Spannability): Every $d_t(p)$ has a finite family of convex subsets $d_t^j(p)$, $j = 1, ..., \kappa_t$ such that

$$d_t(p) = \bigcup_{j=1}^{\kappa_t} d_t^j(p).$$

Assumption 2' (Bounded Nonconvexity): There exits $\epsilon \in \mathbf{R}^{+}$ such that for all $t = 1, ..., T$,

$$\frac{\mathrm{r}(d_t(p))}{\sqrt{n_t}} \leq \epsilon.$$

COROLLARY 1 (NOMURA (1993a, COROLLARY 1), Elementary). *Let* $\mathcal{E} : A \to \mathcal{P}_{mo} \times \mathbf{R}^{+}_{\infty}$ *with* $|A| = n$ *be a "type-decomposable" finite exchange economy with countably many commodities, type-decomposable in the sense that* A *has a disjoint partition* A_t, $t = 1, ..., T$, *the group of consumers of type* t, *where* $|A_t| = n_t$ *and* $\sum_{t=1}^{T} n_t = n$ *such that* $d(p,a) = d_t(p)$ *for all* $a \in A_t$. *Then, there exist*

$$p \in \Delta = \left\{ p \in \mathbf{R}^{+}_{\infty} \middle| \; p. \sum_{a \in A} e(a) \leq 1, p \gg 0 \right\}$$

and a net allocation $g(a) \in \mathsf{con}\, d(p,a)$

Suppose further that $\mathcal{E}$ *satisfies Assumptions* 1' *and* 2'.

Then, for any such g*, there exists a selection* $f(a) \in d(p,a)$ *such that*

$$\frac{1}{n}\left\|\sum_{a\in A} f(a)\right\|_2 \leq \frac{1}{n}\left\|\sum_{a\in A}(f(a)-g(a))\right\|_2 \leq 2\epsilon\sqrt{\sum_{t=1}^{T}\left(\frac{n_t}{n}\right)^2}.$$

In the presence of the nonconvexity due to commodity indivisibility, where natural candidate for the commodity space is

$$\mathbf{R}^d_\infty = \left\{x \in \mathbf{R}^+_\infty \middle|\, x^{i+1} \in \mathbf{N} \cup \{0\} (\forall i \in \mathbf{N})\right\}$$

embodying the minimum requisite of at least one perfectly divisible commodity, say $x^1 \equiv x^h \in \mathbf{R}_+$, the following hypothesis serves to secure the desired upper hemicontinuity of the convex hull of the so-called "Debreu-mapping" whose fixed point characterizes a pair of prices and a net allocation close the the approximate equilibrium.

In order to overcome anomalies caused by the nonconvex commodity space, $\mathcal{P}_{mo}$ needs to be restricted to $\mathcal{P}_{odd}$, where $\succ \in \mathcal{P}_{odd}$ exhibits the *overriding desirability of the perfectly divisible commodity*: $(\forall x \in \mathbf{R}^d_\infty)$ $(\exists \xi$ finite$)[\xi u_1 \succ x]$, where $u_1 = (1, 0, ...) \in \mathbf{R}^d_\infty$.

Furthermore, for those inconvenienced by the commodity indivisibility, $d(p,a)$, *not upper hemicontinuous* any longer, needs to be weakened by incorporating the *satisficing* behavior: Consume so that no superior point is cheaper. Accordingly, define the *weak excess demand set* by

$$d_w(p,a) = \left\{z \middle|\, (z+e(a)) \in \mathbf{R}^d_\infty,\, p.z \leq 0, (\forall y \in \mathbf{R}^d_\infty)\, y \succ_a (z+e(a)) \Rightarrow p.y \geq p.e(a)\right\}.$$

Assumption 3 (Dispersion Hypothesis): Let $e^h(a)$ denote the amount of the perfectly divisible commodity in agent a's initial endowment. For any given countably many values $\alpha_i \in \mathbf{R}_+$, $i \in \mathbf{N}$, these exists $\delta \in [0,1]$ such that

$$\sum_{i\in\mathbf{N}} \left|\left\{a \in A \middle|\, e^h(a) = \alpha_i\right\}\right| \leq \delta.n.$$

Or alternatively, further restrict the preferences to $\mathcal{P}_{idd}$, those that exhibit the *indispensability of the perfectly divisible commodity*: $x, y \in \mathbf{R}^d_\infty$, $x^h = 0$, $y^h > 0 \Rightarrow y \succ x$.

Assumption 4 (Indispensability of the Perfectly Divisible Commodity): For all $a \in A$, $\succ_a \in \mathcal{P}_{idd}$.

Theorem 5 (Nomura (1993a, Theorem 2), Elementary). *Let* $\mathcal{E}^d : A \to \mathcal{P}_{odd} \times \mathbf{R}^d_\infty$ *with* $|A| = n$ *be a finite exchange economy with countably many indivisible commodities,. i.e., an economy constructed with* $\mathbf{R}^d_\infty$ *and consequently* $\mathcal{P}_{odd}$ *in place of* $\mathbf{R}^+_\infty$ *and* $\mathcal{P}_{mo}$*, respectively in the definition of* $\mathcal{E}$*.*

Suppose $\mathcal{E}^d$ *satisfies Assumption* 3*, the Dispersion Hypothesis.*

Then, there exists a subset $E \subset A$ *with* $|E| \geq (1-\delta).n$*, where* δ *is as specified in Assumption* 3*, such that there exist*

$$p \in \Delta = \left\{ p \in \mathbf{R}^+_\infty \middle| \; p. \sum_{a \in A} e(a) \leq 1, p \gg 0 \right\}$$

and a net allocation $g(a) \in \mathsf{con}\, d(p,a)$ *for all* $a \in E$ *and* $g(a) \in \mathsf{con}\, d_w(p,a)$ *for all* $a \in A - E$ *.*

Suppose further that $\mathcal{E}^d$ *satisfies Assumptions* 1 *and* 2*.*

Then, for any such g*, there exists a selection* f *with* $f(a) \in d(p,a)$ *for every* $a \in E$ *and* $f(a) \in d_w(p,a)$ *for every* $a \in A - E$ *such that*

$$\frac{1}{n} \left\| \sum_{a \in A} f(a) \right\|_2 \leq \frac{1}{n} \left\| \sum_{a \in A} (f(a) - g(a)) \right\|_2 \leq \frac{2M}{\sqrt{n}}.$$

In a type-decomposable economy, where the consumers' characteristics are identical within the group of the same type, Assumption 3 takes the following special form:

Assumption 3' (Dispersion Hypothesis): Let e^h_t denote the amount of the perfectly divisible commodity in type t agent's initial endowment. For any given countably many values $\alpha_i \in \mathbf{R}_+$, $i \in \mathbf{N}$, these exists $\delta \in [0,1]$ such that

$$\sum_{i \in \mathbf{N}} \left\{ n_t \middle| \, e^h_t = \alpha_i \right\} \leq \delta.n.$$

Corollary 2 (Nomura (1993a, Corollary 2), Elementary). *Let* $\mathcal{E}^d : A \to \mathcal{P}_{odd} \times \mathbf{R}^d_\infty$ *with* $|A| = n$ *be a "type-decomposable" finite exchange economy with countably many indivisible commodities, i.e.,* A *has a disjoint partition* $\{A_t \subset A | \, t = 1, \ldots, T\}$ *with* $|A_t| = n_t$ *and* $\sum_{i=1}^{T} n_t = n$ *such that* $\succ_a = \succ_t$ *and* $e(a) = e_t$*, and consequently* $d(p,a) = d_t(p)$ *and* $d_w(p,a) = d_{wt}(p)$ *for all* $a \in A_t$*,* $t \in \{1, \ldots, T\}$*.*

Suppose $\mathcal{E}^d$ *satisfies Assumption* 3'*, the Dispersion Hypothesis.*

Then, there exists a subset $\Gamma \subset \{1, \ldots, T\}$ *with* $\sum_{t \in \Gamma} n_t \geq (1-\delta).n$, *where* δ *is as specified in Assumption* 3', *such that there exist* $p \in \Delta$ *and a net allocation* g *with* $g(a) \in \mathsf{con}\, d_t(p)$ *for all* $a \in A_t$, $t \in \Gamma$ *and* $g(a) \in \mathsf{con}\, d_{wt}(p)$ *for all* $a \in A_t$, $t \in \{1, \ldots, T\} - \Gamma$.

Suppose further that $\mathcal{E}$ *satisfies Assumptions* 1' *and* 2'.

Then, for any such g, *there exists a selection* f *with* $f(a) \in d_t(p)$ *for all* $a \in A_t$, $t \in \Gamma$ *and* $f(a) \in d_{wt}(p)$ *for all* $a \in A_t$, $t \in \{1, \ldots, T\} - \Gamma$ *such that*

$$\frac{1}{n} \left\| \sum_{a \in A} f(a) \right\|_2 \leq \frac{1}{n} \left\| \sum_{a \in A} (f(a) - g(a)) \right\|_2 \leq 2\epsilon \sqrt{\sum_{t=1}^{T} \left(\frac{n_t}{n} \right)^2}.$$

Corollary 3 (Nomura (1993a, Corollary 3), Elementary). *Suppose* $\mathcal{E}^d$ *satisfies Assumption* 4, *Indispensability of the Perfectly Divisible Commodity, instead of Assumption* 3, *in addition to Assumptions* 1 *and* 2; *or, in the group-decomposable case, instead of Assumption* 3', *in addition to Assumptions* 1' *and* 2'. *Then, the conclusions of* Theorem 5 *hold with* $E = A$, *and those of* Corollary 2 *with* $\Gamma = \{1, \ldots, T\}$.

3 Core Equivalence and Bargaining Set Equivalence Theorems for Large-Square Economies

Given a finite exchange economy $\mathcal{E}$ as defined in Theorem 4, a *coalition* is a nonempty subset of A. A coalition S can *improve upon* an allocation f if there exists a function $g : S \to \mathbf{R}^+_\infty$ such that $g(a) \succ_a f(a)$ for all $a \in S$, and $\sum_{a \in S} g(a) \leq \sum_{a \in S} e(a)$.

The *core* of $\mathcal{E}$, $\mathcal{C}(\mathcal{E})$, is the set of all allocations which cannot be improved upon by any coalition.

We now give a formal statement of the assumptions, comparable to the preceding *Assumption*s 1 and 2, that enable us to secure the desired average convexity of the per capita better-than set as a subset of $\mathbf{R}_\infty$ to be defined relative the the core allocations.

Assumption 5 (Finite Spannability): Given $f \in \mathcal{C}(\mathcal{E})$, define $\phi(a) = \{x - e(a) \in \mathbf{R}_\infty | \, x \succ_a f(a) (\forall a \in A)\}$. Then, every $\phi(a)$ has a finite family of convex subsets $\phi^j(a)$, $j = 1, ..., \kappa(a)$ of $\phi(a)$ such that

$$\phi(a) = \bigcup_{j=1}^{\kappa(a)} \phi^j(a).$$

Assumption 6 (Bounded Nonconxexity): There exits an $M \in \mathbf{R}^{+}$, such that $\mathrm{r}(\phi(a) \cup \{0\}) \leq M$ for all $a \in A$.

Theorem 6 (Nomura (1992c, Theorem 1), Elementary Core Equivalence). *Let* $\mathcal{E} : A \to \mathcal{P}_{mo} \times \mathbf{R}^{+}_{\infty}$ *with* $|A| = n$ *be a finite exchange economy satisfying* $\sum_{a \in A} e(a) \gg 0$.

Suppose further that $\mathcal{E}$ *satisfies Assumptions* 5 *and* 6.

Then, given $f \in \mathcal{C}(\mathcal{E})$*, there exists* $p \in \left\{ p \in \mathbf{R}^{+}_{\infty} \middle| \; p. \sum_{a \in A} e(a) = 1 \right\}$ *such that*

(1) $\displaystyle \frac{1}{n} \sum_{a \in A} |p. (f(a) - e(a))| \leq \frac{2\sqrt{2M}}{n^{\frac{3}{4}}}$,

(2) $\displaystyle \frac{1}{n} \sum_{a \in A} |\inf \{ p. (f(a) - e(a)) \mid x \succ_a f(a) \}| \leq \frac{2\sqrt{2M}}{n^{\frac{3}{4}}}$.

Given (S, g), an objection to the allocation f, as introduced in the preceding Core Equivalence Theorem, (T, h) is a *counterobjection* to (S, g) if:

(a) $\displaystyle \sum_{a \in T} h(a) \leq \sum_{a \in T} e(a)$;

(b) $h(a) \succ_a g(a)$ for all $a \in T \cap S$, and $h(a) \succ_a f(a)$ for all $a \in T \setminus S$

An objection (S, g) is said to be *justified* if there is no counterobjection to it. The *Mas-Colell Bargaining Set* of $\mathcal{E}$, $\mathcal{B}_M(\mathcal{E})$ is the set of all allocations against which there is no justified objection.

The following assumptions are comparable to the preceding *Assumption*s 1 and 2 (or 1' and 2'), or 5 and 6, and serve to ensure the desired average convexity of the per capita aggregate demand-cum-endowment set as a subset of $\mathbf{R}^{+}_{\infty}$, which was conceived as a natural analogue of Mas-Colell's price characterization.

Assumption 7 (Finite Spannability): Every $D(p, a)$ has a finite family of convex subsets $D^j(p, a)$, $j = 1, ..., \kappa(a)$ of $D(p, a)$ such that

$$D(p, a) = \bigcup_{j=1}^{\kappa(a)} D^j(p, a).$$

Assumption 8 (Bounded Nonconxexity): There exits an $M \in \mathbf{R}^+$, such that $\mathrm{r}\left(D(p,a) \cup \{e(a)\}\right) \leq M$ for all $a \in A$.

Theorem 7 (Nomura (1995, Theorem 1), Elementary Bargaining Set Equivalence). *Let $\mathcal{E} : A \to \mathcal{P}_{mo} \times \mathbf{R}^+_\infty$ with $|A| = n$ be a finite exchange economy satisfying $\sum_{a \in A} e(a) \gg 0$..*

Given $f \in \mathcal{B}_M(\mathcal{E})$, there exists $p \in \left\{ p \in \mathbf{R}^+_\infty \middle| \, p. \sum_{a\in A} e(a) = 1 \right\}$ such that $f(a) \in \mathsf{con}\left(D(p,a) \cup \{e(a)\}\right)$ for all $a \in A$.

Suppose further that $\mathcal{E}$ satisfies Assumptions 7 and 8.

Then, for any such f, there exists a selection $\tilde{f} : a \to \mathbf{R}^+_\infty$ such that

(1) $\tilde{f} \in D(p,a) \cup \{e(a)\}$ *for all* $a \in A$

and

(2) $$\frac{1}{n}\left\| \sum_{a\in A}\left(\tilde{f}(a) - e(a)\right)\right\|_2 \leq \frac{1}{n}\left\| \sum_{a\in A}\left(\tilde{f}(a) - f(a)\right)\right\|_2 \leq \frac{2M}{\sqrt{n}}.$$

4 Average Convexity Theorems of Infinite Dimensional Ranges

For $A \subset \mathbf{R}_\infty$, the *radius* of A is $\mathrm{rad}(A) = \inf_{x\in\mathbf{R}_\infty} \sup_{y \in A} \|x-y\|_2$, and the *inner radius* of A is $\mathrm{r}(A) = \sup_{x \in \mathrm{con}\, A} \inf_{S \subset A \,\mathrm{spans}\, x} \mathrm{rad}(S)$.

Theorem 8 (Nomura (1993a, Lemma 1), Elementary): *Let $G : A \to \mathbf{R}_\infty$ with $|A| = n$.*

Suppose

(i) *every $G(a)$ has a large but finite family of convex subsets G^j, $j = 1, \ldots, \kappa(a)$ of $G(a)$ such that*

$$G(a) = \bigcup_{j=1}^{\kappa(a)} G^j(a),$$

and

(ii) *for some* $M \in \mathbf{R}^{+}$, $\mathrm{r}(G(a)) \le M$ *for all* $a \in A$.

Then, given $y \in \mathsf{con} \sum_{a \in A} G(a)$, *there exists* $x \in \sum_{a \in A} G(a)$ *such that*

$$\|x - y\|_2 \le 2M\sqrt{n}.$$

COROLLARY 4 (NOMURA (1993a, LEMMA 2), Elementary). *Let* $G : A \to \mathbf{R}_\infty$ *with* $|A| = n$.

Suppose A *has a finite disjoint partition* A_i, $i = 1, ..., \rho$ *where* $|A_i| = n_i$ *and* $\sum_{i=1}^{\rho} n_i = n$, *i.e.,* $A = \bigcup_{i=1}^{\rho} A_i$ *and* $A_i \cap A_j = \oslash$ *if* $i \neq j$, *such that* $G(a) = G_i$ *for all* $a \in A_i$

Suppose further that

(i') *every* G_i *has a finite family of convex subsets of such that* G_{ij}, $j = 1, \ldots, \kappa_i$ *such that*

$$G_i = \bigcup_{j=1}^{\kappa_i} G_{ij},$$

and

(ii') *for some* $\epsilon \in \mathbf{R}^{+}$,

$$\frac{\mathrm{r}(G_i)}{\sqrt{n_i}} \le \epsilon$$

for all i.

Then, given $y \in \mathsf{con} \sum_{a \in A} G(a)$, *there exists* $x \in \sum_{a \in A} G(a)$ *such that*

$$\|x - y\|_2 \le 2\epsilon \sqrt{\sum_{i=1}^{\rho} n_i^2}.$$

5 Delineations of My Current Research Agenda

5.1 On the Value Allocations

$v : \mathcal{P}(A) \to \mathbf{R}$, where $\mathcal{P}(A)$ denotes the power set of A, is *superadditive* if, for disjoint subsets $S, T \in \mathcal{P}(A)$, $S \cap T = \oslash$, $v(S \cup T) \ge v(S) + v(T)$. A *finite game with side payments* $\Gamma = (A, v)$ consists of a finite set of agents A and a superadditive function v such that $v(\oslash) = 0$. The *Shapley value* of a finite game Γ assigns to each agent a the expected marginal contribution s_a to all coalitions to which she belongs according to the formula

$$s_a = \sum_{S \subset A} \frac{(|S|-1)!(|A|-|S|)!}{|A|!} \{v(S) - v(S \backslash \{a\})\}.$$

Let $\mathcal{P}$ denote the complete preorders $\succeq$ on $\mathbf{R}_+^k$ that are transitive, continuous, and locally nonsatiated, but not necessarily convex. A *utility function* representing $\succeq \in \mathcal{P}$ is a continuous function $u : \mathbf{R}_+^k \to \mathbf{R}_+$ such that $u(x) \geq u(y) \iff x \succeq y$.

Given a non-atomic exchange economy $\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbf{R}_+^k$ and a family of utility functions $u = \{u_a |\ a \in A\}$, one u_a representing $\succeq_a$, define a game $((A, \mathcal{A}), v_u)$ by

$$v_u(S) = \max \left\{ \int_S u_a(x(a))\nu(da) \middle| \int_S x d\nu = \int_S e d\nu, (\forall a \in S) x(a) \in \mathbf{R}_+^k \right\}.$$

Denote the Shapley value of $((A, \mathcal{A}), v_u)$ by $Sh_a = \lim_{|A|,|S| \to \infty} s_a \Big|_{v(S)=v_u(S)}$, *if it exists at all*! (Indeed, the price characterization of the Shapley value due to Wayne Shafer (1980) will serve as such limit in the subsequent **3.3**.)

An allocation is a map $f : A \to \mathbf{R}_+^k$ such that $\displaystyle\int_A f d\nu = \int_A e d\nu$. Denote by $\mathcal{V}(\mathcal{E})$ *value allocations* for $\mathcal{E}$. An allocation $f \in \mathcal{V}(\mathcal{E}) \overset{\text{def}}{\iff} (\exists \{u_a |\ a \in A\})(\forall a.e.\ a \in A) u_a(f(a)) = Sh_a$.

Let $\mathcal{E}_{con} : (A, \mathcal{A}, \nu) \to \mathcal{P}_{con} \times \mathbf{R}_+^k$ be a non-atomic convex exchange economy. Robert Aumann's equilibrium existence theorem Aumann (1966, Main Theorem) with nonconvex preferences and the convex commodity space, when combined with his value equivalence theorem Aumann (1975, Theorem 1) with *convex* preferences and commodity space, implies $\mathcal{V}(\mathcal{E}_{con}) \neq \oslash$.

Proposition 1 (Aumann (1966, Main Theorem) and (1975, Theorem 1), Measure-Based Value Existence): *Given a non-atomic convex exchange economy* $\mathcal{E}_{con} : (A, \mathcal{A}, \nu) \to \mathcal{P}_{con} \times \mathbf{R}_+^k$, $\mathcal{V}(\mathcal{E}_{con}) \neq \oslash$, *i.e., there exist* $\{u_a |\ a \in A\}$, *an integrable function* $f : A \to \mathbf{R}_+^k$ *and* $S \in \mathcal{A}$ *such that*

(i) $\nu(S) = 1$,

(ii) $(\forall a \in S) u_a(f(a)) = Sh_a$

and

(iii) $\int_A f d\nu \leq \int_A e d\nu$.

PROPOSITION 2 (AUMANN (1975, THEOREM 1), Measure-Based Value Equivalence): *Let* $\mathcal{E}_{con} : (A, \mathcal{A}, \nu) \to \mathcal{P}_{con} \times \mathbf{R}^k_+$ *be a non-atomic convex exchange economy. Then,* $\mathcal{V}(\mathcal{E}_{con}) = \mathcal{W}(\mathcal{E}_{con})$*, i.e., given* $f \in \mathcal{V}(\mathcal{E}_{con})$*, there exist* $p \neq 0$ *and a measurable* $S \subset A$ *such that*

(i) $\nu(S) = 1$;

(ii) $(\forall a \in S) f(a) \in D(p, a)$;

and

(iii) $\int_S f d\nu \leq \int_S e d\nu.$

Let $\Delta = \{p \in \mathbf{R}^k_+ | \ \|p\|_1 = 1\}$. Define the *expenditure function* $M : \mathcal{P} \times \Delta \times \mathbf{R}^k_+ \to \mathbf{R}_+$ by $M(\succeq, p, x) = \min\{p.x' | \ x' \succeq x\}$. If it is continuous, then $M(\succeq, p, x)$ serves as a utility function representing $\succeq$. Indeed,

PROPOSITION 3 (SHAFER (1980)): *Let the expenditure function* $M : \mathcal{P} \times \Delta \times \mathbf{R}^k_+ \to \mathbf{R}_+$ *be defined by* $M(\succeq, p, x) = \min\{p.x' | \ x' \succeq x\}$*. Then,* M *is continuous in* $(\succeq, p, x)$.

For each $p \in \Delta$, consider the game $((A, \mathcal{A}), v(p, .))$ defined by

$$v(p, S) = \max\left\{\int_S M(\succeq_a, p, x(a)) \ \nu(da) \middle| \ x(a) \in \mathbf{R}^k_+, \int_S x \ d\nu = \int_S e \ d\nu\right\}.$$

Then, $v(p, .) : \mathcal{P}(A) \to \mathbf{R}_+$ facilitates the *price characterization of the Shapley value* of $((A, \mathcal{A}), v(p, .))$ as $Sh(p) = \{Sh_a(p) | \ a \in A\}$.

An allocation $f : A \to \mathbf{R}^k_+$ is a value allocation if there exists a $p \in \Delta$ such that $(\forall a.e. \ a \in A) M(\succeq_a, p, f(a)) = Sh_a(p)$. The set of value allocations for $\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbf{R}^k_+$ is now defined, *relative to a choice of* $p \in \Delta$, and will be denoted $\mathcal{V}_p(\mathcal{E})$ in short.

A resort to the price characterization of the Shapley value, and therefore $\mathcal{V}_p(\mathcal{E})$ in place of $\mathcal{V}(\mathcal{E})$, manages to eliminate the convexity requirement on preferences in Aumann's results. A generalization of the followings to admit nonconvex consumption sets $X \subset \mathbf{R}^k_+$ is also straightforward, and will be carried out in the subsequent **5.3.1**.

THEOREM 9 (NOMURA (1992b, THEOREM 3[4]), Measure-Based Value Equivalence): *Given a non-atomic exchange economy* $\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbf{R}^k_+$*, there exists a* $p \in \Delta$ *such that* $\mathcal{V}_p(\mathcal{E}) = \mathcal{W}(\mathcal{E})$*, i.e., given* $p \in \Delta$ *and* $f \in \mathcal{V}_p(\mathcal{E})$*, there exists a measurable* $S \subset A$ *such that*

[4]In NOMURA (1992b), this THEOREM was mixed up with THEOREM 4 which was mislabeled as THEOREM 3, also quoted as THEOREM 11 in the subsequent **5.1.1**. The proof is given in NOMURA (1992b, SECTION 4).

(i) $\nu(S) = 1$;

(ii) $(\forall a \in S)\, f(a) \in D(p, a)$;

and

(iii) $\displaystyle\int_S f d\nu \leq \int_S e d\nu.$

THEOREM 10 (NOMURA (1992b, THEOREM 1), Measure-Based Value Existence): *Given a non-atomic exchange economy $\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbf{R}^k_+$, there exists a $p \in \Delta$ such that $\mathcal{V}_p(\mathcal{E}) \neq \oslash$, i.e., there exist $p \in \Delta$, an integrable function $f : A \to \mathbf{R}^k_+$ and $S \in \mathcal{A}$ such that*

(i) $\nu(S) = 1$,

(ii) $(\forall a \in S)\, M(\succeq_a, p, f(a)) = Sh_a(p)$

and

(iii) $\displaystyle\int_A f d\nu \leq \int_A e d\nu.$

REMARK 1: THEOREM 10 for $\mathcal{V}_p(\mathcal{E})$ generalizes THEOREM 4 of WOODERS and ZAME (1987) for $\mathcal{V}(\mathcal{E})$, by accommodating more than finitely many types and nonconvex preferences. In the light of UNEQUAL TREATMENT THEOREMs due to GREEN (1972) and generalized in KHAN and POLEMARCHAKIS (1978, THEOREMs 1 and 2), their restriction of the core payoffs to the class satisfying the *Equal Treatment Property* is all the more problematic when one attempts at generalizations to non-replica sequences of finite nonconvex exchange economies.

REMARK 2: The outline of the proof is as follows: First prove the hyperfinite limit result *by transfer* of the known finite result, as will be explained in the subsequent **5.1.2**. In the present case, the value convergence result due to WOODERS and ZAME (1987, THEOREM 1) is such finite result for the derived game. After due care taken of the deduction of the results for the hyperfinite economy from those for the derived game, apply the Loeb measure methodology to convert thus obtained hyperfinite result to the standard measure-theoretic counterpart, which will be sketched in the **5.1.3**.

In the subsequent detailed discussions in **5.3.1** on the strength of the hyperfinite approach over the measure-theoretic alternative, THEOREMs 9 and 10 will be cited as *somewhat successful* among the applications of Hyperreal Analysis for the purpose of asymptotoic interpretations, because, as we noted in the previous paragraph, the Transfer Principle, and the Loeb Measure procedure of the Hyperreal Analysis played a central role in deriving the standard measure-theoretic results, which are otherwise unattainable.

5.1.1 The Derived Game

The following definition of a *technology* is an adaptation of the characterization due to WOODERS and ZAME (1987).

Let $\Omega = \mathcal{P} \times X$ be the space of *consumers' characteristics*, where a generic element $\omega = (\succeq, e)$ specifies the consumer attribute in terms of the preferences and the initial endowment. When endowed with the closed convergence topology, the closed nonempty subset $\mathcal{P} \subset \mathcal{P}(X \times X)$ becomes a separable compact metrizable space (HILDENBRAND (1974, **B.II**, THEOREM 2), and let $d_{\succeq}$ be the metric for the topology. Therefore, a metric d_Ω will be given by $d_\Omega(\succeq, \succeq') + \|e - e'\|$, where $\| \ \|$ is the Euclidean norm.

Now, look upon the original finite exchange economy $\mathcal{E}$ as an *attribute function* $\mathcal{E} : A \to \Omega$ with $|A|$ finite. Then, for any subset $S \subset A$, define a *profile* on Ω, $f(\mathcal{E}|S) : \Omega \to \mathbf{N} \cup \{0\}$, by $f(\mathcal{E}|S)(\omega) = |\{a \in A| (\succeq_a, e(a)) = \omega\}|$. The set of profiles on Ω will be denoted by$P(\Omega)$. We write $f(\mathcal{E}|S) = 0$ *iff* $(\exists S \subset A)(\forall \omega \in \Omega)[f(\mathcal{E}|S)(\omega) = 0]$. Without loss of generality, $f(\mathcal{E}|S) = 0$ *iff* $S = \oslash$.

$f(\mathcal{E}|S) \leq f(\mathcal{E}|S')$ *iff* $(\forall \omega \in \Omega)[f(\mathcal{E}|S)(\omega) \leq f(\mathcal{E}|S')(\omega)]$. Note that if $S \subset S'$, then $f(\mathcal{E}|S) \leq f(\mathcal{E}|S')$, but *not vice versa.*

For $\omega \in \Omega$, the profile χ_{ω_0} is such that $\chi_{\omega_0}(\omega) = 0$ if $\omega \neq \omega_0$ and $\chi_{\omega_0}(\omega_0) = 1$.

A *technology* of a *pre-game with consumer attributes* (Ω, Λ) consists of a compact metrizable space of consumer attributes Ω and a function $\Lambda : P(\Omega) \to \mathbf{R}_+$ with the following properties:

(i) $\Lambda(0) = 0$;

(ii) *Superadditivity*: $\Lambda(f(\mathcal{E}|S) + f(\mathcal{E}|S')) \geq \Lambda(f(\mathcal{E}|S)) + \Lambda(f(\mathcal{E}|S'))$;

(iii) *Individual Marginal Bound*: $(\exists M \in \mathbf{R}_+)(\forall \omega \in \Omega)(\forall S \subset A)$
$$[\Lambda(f(\mathcal{E}|S) + \chi_\omega) \leq \Lambda(f(\mathcal{E}|S) + M)];$$

(iv) *Continuity*: $(\forall S \subset A)(\forall \omega, \omega' \in \Omega)(\forall \epsilon > 0)(\exists \delta > 0)$
$$[d_\Omega(\omega, \omega') < \delta \Longrightarrow |f(\mathcal{E}|S) + \chi_\omega) - f(\mathcal{E}|S) + \chi_{\omega'})| < \epsilon].$$

In order to define a finite game with side payments $\Gamma = (A, \nu_{\mathcal{E}})$ as a *derived game* from the technology (Ω, Λ), the *worth* of a coalition S will be given by the characteristic function $\nu_{\mathcal{E}} : 2^{|A|} \to \mathbf{R}_+$ defined by $\nu_{\mathcal{E}}(S) = \Lambda(f(\mathcal{E}|S))$. Then, it is easy to check that $\nu_{\mathcal{E}}$ is superadditive.

The following construction of the nonatomic game $((A, \mathcal{A}, \nu), \xi)$, derived from the technology (Ω, Λ), with ξ to be interpreted as *weights* of coalitions, is patterned after WOODERS and ZAME (1987, SECTION 9):

For $A \in \mathcal{A}$ with $\nu(S) \neq 0$, let $\alpha_t = \dfrac{\nu(S \cap A_t)}{\nu(S)}$ for each t. Then, $\alpha_t \geq 0$ for all t, and $\displaystyle\sum_{t=1}^{T} \alpha_t = 1$

Choose a sequence $\{S_k\}$ of $\mathcal{A}$, such that $|S_k| \to \infty$ (and consequently $|A_k| \to \infty$), and $\dfrac{f(\mathcal{E}|\, S_k)(\omega_t)}{|S_k|} \to \alpha_t$ for each t.

Then, we can define a set function $\xi : \mathcal{A} \to \mathbf{R}_+$, which is interpreted as *weights* of coalitions, such that

$$\xi(S) = \begin{cases} 0 & \text{if } \nu(S) = 0, \\ \left(\lim_{k\to\infty} \frac{\Lambda(f(\mathcal{E}|\, S_k))}{|S_k|}\right) \nu(S) & \text{otherwise.} \end{cases}$$

THEOREM 11 (NOMURA (1982b, THEOREM 4[5])): *Let $((A, \mathcal{A}, \nu), \xi)$ be a nonatomic game derived from the technology (Ω, Λ).*

Then, the Shapley value of $((A, \mathcal{A}, \nu), \xi)$ is in the core, i.e., there exists $p \in \Delta$ such that

(a) *Pareto Optimality*: $Sh(p)(A) = \nu_{\mathcal{E}}(A)$;

and

(b) *For all $S \subset A$, $Sh(p)(S) \geq \nu_{\mathcal{E}}(S)$.*

REMARK 3: When S's reduce to single-element sets, (b) is strengthened to

(c) *Individual Rationality*: For all $a \in A$, $Sh_a(p) \geq \nu_{\mathcal{E}}(\{a\})$,

i.e., the Shapley value coincides with the individually rational core.

5.1.2 Hyperfinite Limit Theorems for Value Allocations

Construct a hyperfinite exchange economy $\mathcal{E}_\omega$ from which the game $(A, \nu_{\mathcal{E}_\omega})$ is derived, on the one hand, and which is convertible to a standard measurable economy $\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbf{R}^k_+$ in THEOREMS 9 and 10 by the Loeb measure methodology, on the other.

[5] Mislabeled as THEOREM 3 in NOMURA (1982b).

Let $(A, \mathcal{I}(A), \lambda)$ be an *internal hyperfinite space of agents*, where A with $|A| = \omega \in {}^*\mathbf{N} - \mathbf{N}$ is endowed with an *internal subset algebra* $\mathcal{I}(A)$, and an *internal counting measure* $\lambda = \dfrac{|S|}{|A|}$ for all $S \in \mathcal{I}(A)$, which is easily checked to be an *internal finitely additive infinitesimal measure.*

A *hyperfinite exchange economy* is an *internal* $\mathcal{I}(A)$*-measurable* map $\mathcal{E}_\omega : (A, \mathcal{I}(A), \lambda) \to {}^*\mathcal{P} \times {}^*\mathbf{R}_+^k$.

By transfer of Wooders and Zame (1987, Theorem 1), we have:

Theorem 12 (Nomura (1982b, Theorem 5.2), Hyperfinite Limit Theorem in the Derived Game): *Let* $(A, \nu_{\mathcal{E}_\omega})$ *with* $|A| = \omega \in {}^*\mathbf{N} - \mathbf{N}$ *be a hyperfinite game obtained as the nonstandard extension of* $(A, \nu_{\mathcal{E}})$ *that is derived from the technology* (Ω, Λ).

Then, the Shapley value of $(A, \nu_{\mathcal{E}_\omega})$ *is in the core, i.e., there exists* $p \in {}^*\Delta$ *such that*

(a) *Pareto Optimality*: $Sh(p)(A) = \nu_{\mathcal{E}_\omega}(A)$;

and

(b) *For all internal subsets* $S \subset A$, $(1/|S|)\,(Sh(p)(S) - \nu_{\mathcal{E}_\omega}(S)) \gtrsim 0$.

Remark 4: When S's reduce to single-element sets, (b) is strengthened to

(c) *Individual Rationality*: For all $a \in A$, $Sh_a(p) \gtrsim \nu_{\mathcal{E}_\omega}(\{a\})$.

The following Theorem 13 for the hyperfinite exchange economy $\mathcal{E}_\omega$ is deduced from the preceding Theorem 12 by construction of the derived game $(A, \nu_{\mathcal{E}_\omega})$ from $\mathcal{E}_\omega$.

Theorem 13 (Nomura (1982b, Theorem 5.1), Hyperfinite Limit Theorem in the Original Economy): *Let* $\mathcal{E}_\omega : (A, \mathcal{I}(A), \lambda) \to {}^*\mathcal{P} \times {}^*\mathbf{R}_+^k$ *be a hyperfinite exchange economy with* $|A| = \omega \in {}^*\mathbf{N} - \mathbf{N}$.

Then, there exists a value allocation in $\mathcal{E}_\omega$*, i.e., there exist* $p \in {}^*\Delta$*, an S-integrable function* $f : A \to {}^*\mathbf{R}_+^k$ *and* $S \in \mathcal{I}(A)$ *such that*

(i) $\lambda(S) = 1$;

(ii) *For all* $a \in S$, $M(\succ_a, p, f(a)) = Sh_a(a)$;

and

(iii) $\sum_{a \in A} f(a)\lambda(a) \lesssim \sum_{a \in A} e(a)\lambda(a)$.

Remark 5: A generalization of Theorems 9 and 10 to accommodate the nonconvex commodity space $X \subset \mathbf{R}_+^k$ is straightforward with a resort to an appropriate version of the *Dispersion Hypothesis* (*D.H.*). (*D.H.*)'s for $\mathcal{E}_\omega$ and $(A, \nu_{\mathcal{E}_\omega})$ are conceived with asymptotic interpretations in mind for finite economies $\mathcal{G} = \{\mathcal{E}_n\}$, and the derived games thereof, and therefore, they are the *-transform of (*D.H.*) in $\mathcal{E}_n$ and $(A, \nu_{\mathcal{E}_n})$, respectively.

(*D.H.*) *for* $\mathcal{G} = \{\mathcal{E}_n\}$ *with nonconvex* $X \subset \mathbf{R}_+^k$ (Nomura (1982b, Theorem 2)): For all $\mathcal{E}_n \in \mathcal{G}$, and for all $(p, x) \in \Delta \times \mathrm{con}X$, and for all $\alpha \in \mathbf{R}_+$, there exists an $M \in \mathbf{N}$ such that

$$|\{a \in A|\, M(\succeq_a, p, x) = \alpha\}| \leq M.$$

(*D.H.*) *for* $\{(A, \nu_{\mathcal{E}_n}))\}$ *with nonconvex* $X \subset \mathbf{R}_+^k$ (Dispersion with respect to consumer attributes): For each $a \in A$ and for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\{a' \in A|\, d_\Omega\left((\succeq_a, e(a)), (\succeq_{a'}, e(a'))\right) < \epsilon\}| < \delta|A|.$$

Thus, (*D.H.*) in each context reads as:

(*D.H.*) *for* $\mathcal{E}_\omega$ *with nonconvex* $X \subset {}^*\mathbf{R}_+^k$ (Nomura (1982b, Theorem 5.1); Also quoted as Theorem 13 in the above): For all $(p, x) \in^* \Delta \times \mathrm{con}^*X$, and for all $\alpha \in {}^*\mathbf{R}_+$,

$$\lambda\left(\{a \in A|\, M(\succeq_a, p, x) = \alpha\}\right) \simeq 0$$

(*D.H.*) *for* $\mathcal{E} = \mathrm{st}(\mathcal{E}_\omega)$ *with nonconvex* $X \subset \mathbf{R}_+^k$ (Nomura (1982b, Theorem 1); Also quoted as Theorem 10 in **5.1**): For all $(p, x) \in \Delta \times \mathrm{con}X$, and for all $\alpha \in \mathbf{R}_+$,

$$\mu_M(\alpha) = \nu\left(\{a \in A|\, M(\succeq_a, p, x) = \alpha\}\right) = 0,$$

where $\mu_M(\alpha)$ denotes the *utilities distribution*, generated from ν.

(D.H.) for $(A, \nu_{\mathcal{E}_\omega})$ *with nonconvex* $X \subset {}^*\mathbf{R}^k_+$ (NOMURA (1982b, THEOREM 5.2); Also quoted as THEOREM 12 in the above): For each $a \in A$,

$$\frac{|\{a' \in A|\, d_\Omega\left((\succeq_a, e(a)), (\succeq_{a'}, e(a'))\right) \simeq 0\}|}{|A|} \simeq 0.$$

(D.H.) for $\left(A, \nu_{\mathrm{st}(\mathcal{E}_\omega)}\right)$ *with nonconvex* $X \subset \mathbf{R}^k_+$ (NOMURA (1982b, THEOREM 4[6]); Also quoted as THEOREM 11 in **5.1.1**): For each $a \in A$,

$$\mu_\Omega(a) = \nu\left(\{a' \in A|\, d_\Omega\left((\succeq_a, e(a)), (\succeq_{a'}, e(a'))\right) = 0\}\right) = 0.$$

5.1.3 Conversion of Hyperfinite Results to Standard Measure-Theoretic Counterparts

In order to convert $\mathcal{E}_\omega$ into the standard measurable economy, we apply the Loeb measure methodology, by looking upon the set of agents A, now as a standard infinite set. Consumption characteristics are derived as those associated with functionals on Loeb spaces.

We start with the conversion of the measurable space of agents. Let $(A, \mathcal{A}, \nu)$.be constructed as the Loeb measure space of $(A, \mathcal{I}(A), \lambda)$, denoted by $(A, L(\mathcal{I}(A)), L(\lambda))$, where $L(\mathcal{I}(A))$ is the completion of $\sigma(\mathcal{I}(A))$, the smallest σ-algebra containing $\mathcal{I}(A)$, and $L(\lambda)$ is the unique extension of $\mathrm{st}(\lambda)$ to $\sigma(\mathcal{I}(A))$. By PROPOSITION 7 in the subsequent **5.3.3**, ν is a countably additive non-atomic measure.

A standard Loeb measure economy $\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbf{R}^k_+$ is constructed as $\mathrm{st}(\mathcal{E}_\omega)$, the standard part map of $\mathcal{E}_\omega$. Together with the following properties, which are immediate consequences of the Loeb measure spaces, the Loeb measure economy thus constructed indeed qualifies as the standard measurable economy for which THEOREMs 9 and 10 hold.

1. A standard Loeb measure exchange economy $\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbf{R}^k_+$, constructed as $\mathrm{st}(\mathcal{E}_\omega)$, is $\mathcal{A}$-measurable.
2. $\mathcal{P}$ is compact in the topology of closed convergence.
3. $e(a)$ is ν-integrable.

[6]Mislabeled as THEOREM 3 in NOMURA (1982b).

5.2 Approximate Decentralization of Bargaining Sets

5.2.1 Alternative Characterizations of Bargaining Sets

- *Aumann-Maschler Bargaining Set*: The *Aumann-Maschler bargaining set* of $\mathcal{E}$, $\mathcal{B}_{AM}(\mathcal{E})$, introduced in AUMANN and MASCHLER (1964, Definition 2.2.(20)), is a special case of the subsequent Geanakoplos δ-bargaining set $\mathcal{B}_\delta(\mathcal{E})$ restricting U to a single "leader", i.e.,

$$\mathcal{B}_{AM}(\mathcal{E}) = \mathcal{B}_{\frac{1}{|A|}}(\mathcal{E}).$$

- *Geanakoplos Bargaining Set*: GEANAKOPLOS (1978, Section VI) modifies the Aumann-Maschler characterization by requiring that an objection be proposed by a group U of leaders, consisting of a fixed small fraction δ of the participants in the initial objection, who are to refrain from participating in a counterobjection coalition T (note well the requirement $T \cap U = \oslash$).

  Let $\mathcal{E} : A \to \mathcal{P} \times L_+$ be a finite exchange economy such that $|A| \in \mathbf{N}$. (S, U, g) is a *δ-objection* to an allocation f.3 if there exists a coalition, i.e., a nonempty subset $S \subset A$, a set of "leaders" $U \subset S$ with $0 \leqq \frac{|U|}{|S|} \leqq \delta$, and $g : S \to L_+$ such that

  (a) $\sum_{a \in S} g(a) \leqq \sum_{a \in S} e(a)$; and

  (b) $g(a) \succeq_a f(a)$ for all $a \in S$ with strict preference for at least one $a \in S$.

  (T, h) is a counterobjection to (S, U, g) if there exist a nonempty $T \subset A$ with $T \cap U = \oslash$, and $h : T \to L_+$ such that

  (c) $\sum_{a \in T} h(a) \leqq \sum_{a \in T} e(a)$; and

  (d) (i) $h(a) \succ_a g(a)$ for all $a \in T \cap S$, (ii) $h(a) \succ_a f(a)$ for all $a \in T \setminus S$.

  A δ-objection is *justified* if there is no counterobjection to it.

  The *Geanakoplos δ-bargaining set* of $\mathcal{E}$, denoted by $\mathcal{B}_\delta(\mathcal{E})$, is the set of all allocations to which every δ-objection has a counterobjection.

The *Geanakoplos bargaining set* $\mathcal{B}_G(\mathcal{E})$ is expressed as

$$\mathcal{B}_G(\mathcal{E}) = \bigcup_{\delta \in [0,1]} \mathcal{B}_\delta(\mathcal{E}).$$

- *Zhou Bargaining Set*: Zhou (1994, Definition 2.2) downsizes the Mas-Colell bargaining set $\mathcal{B}_M(\mathcal{E})$ by adding following restrictions on counterobjections to define the *Zhou bargaining set* $\mathcal{B}_Z(\mathcal{E})$:

  $$(1) S \not\subseteq T; (2) T \not\subseteq S; \textit{ and } (3) T \cap S \neq \oslash.$$

  In particular, the "nonempty intersection property" (3) requires that a counterobjection be formed from within an objecting coalition to void the initial objection. A counterobjection (S, g) exercises a deterrent effect on $T \cap S \neq \oslash$ from proposing the objection (S, g) in the first place.

5.2.2 Known Results on Convergence of Bargaining Sets

TABLE 1
(Non-)Convergence of Various Bargaining Sets

Bargaining Sets	Anderson (1998)	Anderson, Trockel and Zhou (1997)
Aumann-Maschler B.S. $\mathcal{B}_{AM}(\mathcal{E})$	Convergence (Theorem 4.4)	
Geanakoplos B.S. $\mathcal{B}_\delta(\mathcal{E})$	Convergence (Theorems 3.4 and 3.6)	
Mas-Colell B.S. $\mathcal{B}_M(\mathcal{E})$		Non-convergence (Theorem 3.3)
Zhou B.S. $\mathcal{B}_Z(\mathcal{E})$		Non-convergence (Theorem 3.3)

- Anderson (1998, Theorem 4.4) establishes convergence of $\mathcal{B}_{AM}(\mathcal{E})$ for $\mathcal{E}$ with $\mathbf{R}^k_+$ as the commodity space (and with further restricted preferences on $\mathbf{R}^k_+$ than usual).

- Anderson (1998, Theorems 3.4 and 3.6) establish convergence of $\mathcal{B}_\delta(\mathcal{E})$ for $\mathcal{E}$ with the commodity space $\mathbb{R}^k_+$.

- Certainly, $\mathcal{B}_Z(\mathcal{E}) \subset \mathcal{B}_M(\mathcal{E})$. Anderson, Trockel and Zhou (1997, Theorem 3.3) show that $\mathcal{B}_Z(\mathcal{E})$ needs not converge for replica sequences $\{\mathcal{E}_r\}$ with the commodity space $\mathbb{R}^k_+$.

5.2.3 Salvaging by "Modified" Convergence of Bargaining Sets

An atomless exchange economy is a measurable map $\tilde{\mathcal{E}} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbf{R}^k_+$ such that $0 \ll \int_A ed\nu \ll \infty$.

Among the many characterizations of bargaining sets, consider in particular the *Mas-Colell bargaining set* of $\mathcal{E}$, $\mathcal{B}_M(\tilde{\mathcal{E}})$ consisting of all allocations to which there is no justified weak objection, i.e., to which every weak objection has a weak counterobjection.

The objection (S, g) to the allocation f is *Walrasian* if there exists $p \neq 0$ such that

(i) $(\forall a.e.\ a \in S)(\forall y \in \mathbf{R}^k_+) y \succeq_a g(a) \Rightarrow p.y \geq p.e(a)$;

and

(ii) $(\forall a.e.\ a \in A \backslash S)(\forall y \in \mathbf{R}^k_+) y \succeq_a f(a) \Rightarrow p.y \geq p.e(a)$.

Conditions (i), (ii) may be rewritten as:

(i') $(\forall a.e.\ a \in S) p.g(a) \geq p.e(a) \Rightarrow g(a) \in D(p, a)$;

and

(ii') $(\forall a.e.\ a \in A \backslash S) p.f(a) \geq p.e(a) \Rightarrow f(a) \in D(p, a)$.

(That is, Walrasian objections possess a self selection property in that S is formed by those who, at p, would rather demand g than f.)

Proposition 4 (Mas-Colell (1989, Proposition 1), Optimality Lemma): *Any Walrasian objection (S, g) to an allocation f is justified.*

Proposition 5 (Mas-Colell (1989, Proposition 2), Existence Lemma): *Suppose an allocation $f \notin \mathcal{W}(\tilde{\mathcal{E}})$. Then, there exists a Walrasian objection (S, g) to f.*

Combining these two Lemmata yields the following equivalence theorem.

Proposition 6 (Mas-Colell (1989, Theorem 1), Measure-Based Bargaining Set Equivalence): *Let a measurable map $\tilde{\mathcal{E}} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbf{R}^k_+$ be an atomless exchange economy such that $0 \ll \int_A ed\nu \ll \infty$. Then, $\mathcal{B}_M(\tilde{\mathcal{E}}) = \mathcal{W}(\tilde{\mathcal{E}})$, i.e., given $f \in \mathcal{B}_M(\tilde{\mathcal{E}})$, there exist $p \neq 0$ and a measurable $S \subset A$ such that*

(i) $\nu(S) = 1$;

(ii) $(\forall a \in S) f(a) \in D(p, a)$;

and

(iii) $\displaystyle\int_S f d\nu \leq \int_S e d\nu.$

For a finite exchange economy $\mathcal{E} : A \to \mathcal{P} \times \mathbb{R}^k_+$, the following Theorem 14 on the "modified" convergence of the Mas-Colell bargaining set salvages the nonconvergence property as exemplified by a counterexample due to Anderson, Trockel and Zhou (1997, Example 3.1 and Theorem 3.3). Instead, Theorem 14 asserts "near-"equivalence of the Mas-Colell bargaining set and the "modified-"Walrasian assignments in the following specific sense:

- "Modified-" in the sense that each individual demand set is enlarged to its union with her initial endowment point;

and

- "near-" in the sense that the modified-Walrasian assignments are deviant from the "near-"modified-Walrasian allocations that serve to decentralize the Mas-Colell bargaining set allocations with the deviations measured by error terms convergent to 0 as the size of the economy increases.

In short, Theorem 14 claims that a Mas-Colell bargaining set allocation can be priced out as a near-modified-Walrasian allocation, which in turn may be approximated by a modified-Walrasian assignment.

Let $M = \max\{\|e(a)\|_1 \mid a \in A\}$ be the bound defined independently of agents' preferences. Given the Mas-Colell bargaining set allocation f, it is always possible to find prices p supporting f as a near-modified-Walrasian allocation, i.e., chosen from $\operatorname{con}(D(p,a) \cup \{e(a)\})$, which will further be approximated within $\dfrac{kM}{\sqrt{n}}$ by a modified-Walrasian selection $\tilde{f}$ from the exact demand or her endowment, so that the degree of non-competitiveness as measured by the "allocation-likeness" of $\tilde{f}$, i.e., the per capita aggregate divergence of the nearly decentralizing modified-Walrasian *assignment* from the original Mas-Colell bargaining set *allocation*, is bounded in norm by $\dfrac{(k+1)M}{\sqrt{n}}$.

Theorem 14 (Nomura (1998a, Theorem), "Modified" Convergence): *Let $\mathcal{E} : A \to \mathcal{P} \times \mathbf{R}^k_+$ be a finite exchange economy with $|A| = n$, and* $M = \max\{\|e(a)\|_1 : a \in A\}$. Denote $\Delta = \left\{p \in \mathbf{R}^l_+ \mid \|p\|_\infty \leq 1, p \gg 0\right\}$.

Suppose $f \in \mathcal{B}_M(\mathcal{E})$. Then, there exists $p \in \Delta$ such that

$$f(a) \in \operatorname{con}(D(p,a) \cup \{e(a)\}) \text{ for all } a \in A.$$

Furthermore, for any such f, there exists a selection $\tilde{f} : A \to \mathbf{R}^k_+$ such that

(i) $\tilde{f}(a) \in D(p,a) \cup \{e(a)\}$ for all $a \in A$;

and

(ii)

$$\frac{\sum_{i=1}^{k} \max\left\{\sum_{a\in A}\left(\tilde{f}^i(a) - f^i(a)\right), 0\right\}}{n} \leqq \frac{kM}{\sqrt{n}},$$

$$\frac{\sum_{i=1}^{k} \max\left\{\sum_{a\in A}\left(\tilde{f}^i(a) - e^i(a)\right), 0\right\}}{n} \leqq \frac{(k+1)M}{\sqrt{n}}.$$

Sketch of the Proof: The proof is patterned after Mas-Colell (1989, Sections 3 and 4) for a measure space of agents and a finite number of commodities. The proof consists of Lemmata 1 and 2, and the contrapositive of the combined assertion of the two yields the desired result: *If there is no modified-Walrasian objection to the selection $\tilde{f}$, within a distance convergent to* 0 *from a Mas-Colell bargaining set allocation $f \in \mathcal{B}_M(\mathcal{E})$ due to the convexification involved, then $\tilde{f}$ is modified Walrasian.*

Let (S, g) be a near-modified Walrasian objection to an allocation f, i.e., such that for all $a \in S$, $p.g(a) \leq p.e(a) \Longrightarrow g(a) \in \mathrm{con}(D(p,a) \cup \{e(a)\})$. Let (S, g) be further approximated by a modified-Walrasian objection $(S, \tilde{g})$ for which $p.\tilde{g} \leq p.e(a) \Longrightarrow g(a) \in (D(p,a) \cup \{e(a)\}$ for all $a \in S$ with error terms satisfying

$$\frac{\sum_{i=1}^{k} \max\left\{\sum_{a\in S}\left(\tilde{f}^i(a) - f^i(a)\right), 0\right\}}{|S|} \leqq \frac{kM}{\sqrt{n}},$$

$$\frac{\sum_{i=1}^{k} \max\left\{\sum_{a\in S}\left(\tilde{f}^i(a) - e^i(a)\right), 0\right\}}{|S|} \leqq \frac{(k+1)M}{\sqrt{n}},.$$

where $M = \max\{\|e(a)\|_1 \mid a \in A\}$.

1. Lemma 1 asserts that *any modified Walrasian objection (S, g) is justified.*

2. Lemma 2 asserts that *if f is not a near-modified Walrasian allocation, then there is modified-Walrasian objection $(S, \tilde{g})$, by way of a near-modified-Walrasian objection (S, g) within the same per capita divergence terms as specified in* Lemma 1.

LEMMA 1 (NOMURA(1998, LEMMA 1), OPTIMALITY LEMMA): *Given an allocation$f$, let (S, g) be a near-modified Walrasian objection to f, i.e., nonempty $S \subset A$ and $g : S \to \mathbf{R}^k_+$ be such that*

(i) $\sum_{a \in S} g(a) \leq \sum_{a \in S} e(a)$,

and

(ii) *$g(a) \succeq_a f(a)$ for all $a \in S$, and $g(a) \succ_a f(a)$ for some $a \in S$,*

and for some $p \neq 0$

(iii) *for all $a \in S$, $p.g(a) \leq p.e(a) \Longrightarrow g(a) \in \text{con}\,(D(p, a) \cup \{e(a)\})$, and*

(iv) *for all $a \in A \setminus S$, $p.f(a) \leq p.e(a) \Longrightarrow g(a) \in D(p, a) \cup \{e(a)\}$.*

Given such objection to (S, g) to f, let $(S, \tilde{g})$ be a modified -Walrasian objection, i.e., such that, in addition to (ii) *and* (iv), (i) *is approximated as*

(1)

$$\frac{\sum_{i=1}^{k} \max\left\{\sum_{a \in S}\left(\tilde{f}^i(a) - f^i(a)\right), 0\right\}}{|S|} \leqq \frac{kM}{\sqrt{n}},$$

and

$$\frac{\sum_{i=1}^{k} \max\left\{\sum_{a \in S}\left(\tilde{f}^i(a) - e^i(a)\right), 0\right\}}{|S|} \leqq \frac{(k+1)M}{\sqrt{n}},.$$

where $M = \max\{\|e(a)\|_1 \mid a \in A\}$,

and (iii) *is strengthened to*

(2) *for all $a \in S$, $p.\tilde{g}(a) \leq p.e(a) \Longrightarrow \tilde{g}(a) \in D(p, a) \cup \{e(a)\}$.*

Let $T \subset A$ and $h : T \to \mathbf{R}^k_+$. Then, for any (T, h), either

(a) $\sum_{a \in T} h(a) > \sum_{a \in T} e(a)$,

or

(b) *$\tilde{g}(a) \succeq_a h(a)$ for some $a \in T \cap S$, or $f(a) \succeq_a h(a)$ for some $a \in T \setminus S$.*

LEMMA 2 (NOMURA(1998, LEMMA 2), EXISTENCE LEMMA): *Suppose an allocation f is such that, for any $p \neq 0$ and for some $a \in A$, $f(a) \notin$* $\text{con}\,(D(p, a) \cup \{e(a)\})$

Then, given such f, there exists a near-modified-Walrasian objection (S, g) to f, i.e., nonempty $S \subset A$ and $g : S \to \mathbf{R}^k_+$ such that

(i) $\sum_{a \in S} g(a) \leq \sum_{a \in S} e(a)$,

and

(ii) $g(a) \succeq_a f(a)$ *for all* $a \in S$, *and* $g(a) \succ_a f(a)$ *for some* $a \in S$,

and for some $p \neq 0$

(iii) *for all* $a \in S$, $p.g(a) \leq p.e(a) \Longrightarrow g(a) \in \operatorname{con}(D(p,a) \cup \{e(a)\})$, *and*

(iv) *for all* $a \in A \setminus S$, $p.f(a) \leq p.e(a) \Longrightarrow g(a) \in D(p,a) \cup \{e(a)\}$.

Furthermore, for any such g, *there exists a modified-Walrasian selection* $\tilde{g}(a) \in D(p,a) \cup \{e(a)\}$*such that, in addition to* (ii) *and* (iv), (i) *is approximated as*

(1)

$$\frac{\sum_{i=1}^{k} \max\left\{\sum_{a \in S}\left(\tilde{f}^{i}(a) - f^{i}(a)\right), 0\right\}}{|S|} \leqq \frac{kM}{\sqrt{n}},$$

and

$$\frac{\sum_{i=1}^{k} \max\left\{\sum_{a \in S}\left(\tilde{f}^{i}(a) - e^{i}(a)\right), 0\right\}}{|S|} \leqq \frac{(k+1)M}{\sqrt{n}},.$$

where $M = \max\{\|e(a)\|_1 \mid a \in A\}$,

and (iii) *is strengthened to*

(2) *for all* $a \in S$, $p.\tilde{g}(a) \leq p.e(a) \Longrightarrow \tilde{g}(a) \in D(p,a) \cup \{e(a)\}$.

Remark 6: It is worth emphasizing that the present "modified" convergence theorem highlights and leaved unanswered the compositional question as to which subgroups of agents in a fixed finite economy are actually assigned to their initial endowment in the modified Walrasian assignment.

Remark 7: In light of the Mas-Colell's Equivalence Theorem for a continuum economy, the set of near-modified Walrasian allocations needs to be checked to be weakly convergent to the set of Walrasian allocations.

- "Nearness" due to the convexification will be eliminated in the limit. Intuitively, this should follow follow in the limit from the Lyapunov Theorem which replaces the Shapley-Folkman Theorem in our Existence Lemma.

- "Modified": Construct the sequence of finite economies $\{\mathcal{E}_n\}$ in an obvious manner by identifying our $\mathcal{E}$ as its n-th element. Denote by $\mathcal{W}_{\text{mod}}(\mathcal{E}_n)$ the set of modified Walrasian allocations in $\mathcal{E}_n$. It is yet to be checked if $\left\{a \in A_n |\, \tilde{f}_n(a) = e_n(a), \tilde{f}_n \in \mathcal{W}_{\text{mod}}(\mathcal{E}_n)\right\}$ is weakly convergent to a measure 0 subset of A, the limit of A_n with a slight notational abuse.

REMARK 8: Presence of a group of "leaders" *à la* Aumann-Davis-Maschler $\mathcal{B}_{AM}(\mathcal{E}_n)$ (AUMANN and MASCHLER (1964), and in particular DAVIS and MASCHLER (1967 [1963]), or GEANAKOPLOS (1978) $\mathcal{B}_G(\mathcal{E}_n) = \bigcup_{\delta \in [0,1]} \mathcal{B}_\delta(\mathcal{E}_n)$ stipulating a fixed proportion δ of any objection, for any $\delta \in [0,1]$, being leaders who refrain from proposing any counterobjection to the objection in question: Compare our "MODIFIED" CONVERGENCE THEOREM with CONVERGENCE THEOREMS for $\mathcal{B}_{AM}$ and $\mathcal{B}_G$ due to ANDERSON (1998, THEOREMS 3.4 and 3.7), as summarized in TABLE 1 in the preceding **5.2.1**.

1. Will the stiplulated presence of leaders ensure

$$\left| \left\{a \in A_n |\, \tilde{f}_n(a) = e_n(a), \tilde{f}_n \in \mathcal{W}_{\text{mod}}(\mathcal{E}_n)\right\} \right| / |A_n| \to 0 \text{ ?}$$

In other words, does the presumption of leaders ensure the Mas-Colell bargaining set allocation $\mathcal{B}_M(\mathcal{E}_n)$ to be realized as a result of almost everyone pursuing actively to attain her demand in a market so that those exceptional ones not actively participating in a market

$$\left\{a \in A_n |\, \tilde{f}_n(a) = e_n(a), \tilde{f}_n \in \mathcal{W}_{\text{mod}}(\mathcal{E}_n)\right\}$$

typically constitute a "finite" subset throughout, and consequently remain to be *negligible* "in the limit" ?

2. Although the targets of convergence are different, a careful comparison needs be made between the convergence speeds, $O(\sqrt{n})$ we found for the Mas-Colell Bargaining Set to the modified Walrasian assignments, and $O(n)$ due to ANDERSON (1998, PROPOSITION 3.10) for the Geanakoplos Bargaining Set to The Walrasian allocations.

REMARK 9: It remains as our conjecture that nonconvergence of the Zhou Bargaining Set $\mathcal{B}_Z(\mathcal{E}_n)$ due to ANDERSON, TROCKEL and ZHOU (1997) may be salvaged as yet another "modified" convergence to an accordingly enlarged set replacing the set of Walrasian allocations.

REMARK 10: It is a straightforward generalization to incorporate the nonconvex commodity space in our elementary convergence theorem, and establish a finite analogue of YAMAZAKI's EQUIVALENCE THEOREM (1993) by weakening both the Mas-Colell Bargaining Set and modified-Walrasian allocations, as prescribed in YAMAZAKI (1995). Such an extension was carried out in the present author's presentation (NOMURA (1995)) under the generality with infinitely many indivisible commodities, where the proof of the EXISTENCE LEMMA was adapted from NOMURA (1993a).

5.3 On the Strengths of the Hyperfinite Approaches

5.3.1 Asymptotic Interpretations

Step 1: Construct a *purely competitive* sequence of finite economies $\mathcal{G} = \{\mathcal{E}_n | \, n \in \mathbf{N}\}$, and idealize the limit of $\mathcal{G}$ by the hyperfinite economy.

Step 2: Prove the desired property for the limit hyperfinite economy $\mathcal{E}_\omega$.

Step 3: Deduce the desired asymptotic result from the limit result proven in *Step 2* for $\mathcal{E}_\omega$.

According to the characterization given in BROWN and KHAN (1980), $\mathcal{G} = \{\mathcal{E}_n | \, n \in \mathbf{N}\}$ is said to be *purely competitive* if

(i) $|A_n| \to \infty$ as $n \to \infty$,

(ii) $\displaystyle\lim_{n\to\infty} \frac{1}{|A_n|} \sum_{a \in A_n} e(a)$ exists,

(iii) $E_n \subset A_n$ and $\displaystyle\lim_{n\to\infty} \frac{|E_n|}{|A_n|} = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{|A_n|} \sum_{a \in E_n} e(a) = 0$,

(iv) For all $\delta > 0$, there exists a compact $K \subset \mathcal{P}$ in the topology of closed convergence, and $\bar{n} \in \mathbf{N}$ such that for all $n \in \mathbf{N}$, $n \geq \bar{n}$ implies $\dfrac{|\{a_n \in A_n | \succ_a \in K\}|}{|A_n|} \geq 1 - \delta$

The corresponding hyperfinite economy $\mathcal{E}_\omega$ idealizing the limit of $\mathcal{G}$ is constructed by transfer from the constituent $\mathcal{E}_n$, n finite, and will be endowed with an internal infinitesimal measure structure. That is, $(A, \mathcal{I}(A), \lambda)$ is an internal infinitesimal measure space of agents, where A with $|A| = \omega \in {}^*\mathbf{N} - \mathbf{N}$ is endowed with an internal subsets algebra $\mathcal{I}(A)$, and an internal counting measure $\lambda = \dfrac{|B|}{|A|}$ for all $B \in \mathcal{I}(A)$, which is easily checked to be an internal finitely additive infinitesimal measure.

Consider $(A, \sigma(\mathcal{I}(A)), L(\lambda))$ the Loeb measure space of $(A, \mathcal{I}(A), \lambda)$, i.e., $\sigma(\mathcal{I}(A))$ is the smallest σ-algebra containing $\mathcal{I}(A)$, and $L(A)$ is the unique extension of $\mathrm{st}(\lambda)$ to $\sigma(\mathcal{I}(A))$. By Proposition 8 (Rashid (1979, Lemma 1)) in the subsequent **5.3.3**, ν is a countably additive non-atomic measure.

A standard measurable economy $\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times X$ will be constructed as $\mathrm{st}(\mathcal{E}_\omega)$, the standard part map of $\mathcal{E}_\omega$.

The followings are immediate consequences of the Loeb measure construction.

Lemma 3 (Nomura (1984, Lemma 13)): *$\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times X$, constructed as $\mathrm{st}(\mathcal{E}_\omega)$, is $\mathcal{A}$-measurable.*

Lemma 4 (Nomura (1984, Lemma 14)): *$\mathcal{P}$ is compact in the topology of closed convergence.*

Lemma 5 (Nomura (1984, Lemma 15)): *$e(a)$ is ν-integrable.*

Depending on the way how *Step 2* is carried out, there are several types of hyperfinite limit theorems with different convincing power:

- **Successful Vindication of the Strength of the Hyperreal Approach over the Continuum Approach:** Asymptotic interpretations of hyperfinite results proven *ab initio*, or "from the scratch" in *Step 2* serve this purpose.

  The hyperfinite equilibrium existence theorem with the commodity space ${}^*\mathbf{R}_\infty$ (Theorem 2 (Nomura (1981), Also quoted as Theorem 2 in Section **2** of the present article) is the case in point.

  This hyperfinite limit result is proven "from the scratch" in two steps: First, prove the equilibrium existence with ${}^*\mathbf{R}^\nu$, $\nu \in {}^*\mathbf{N} - \mathbf{N}$, the ν-truncation of ${}^*\mathbf{R}_\infty$ as the commodity space (Theorem 3 (Nomura (1981), quoted as Theorem 1 in Section **2**), *by transfer* of the equilibrium existence result for a comparable finite economy with a finite number of commodities, say $\mathbf{R}^k$, $k \in \mathbf{N}$. Then, establish the *extensibility* of the result through the transition from the partial ${}^*\mathbf{R}^\nu$ to the full fledged ${}^*\mathbf{R}_\infty$ as the commodity space, by checking the addition of the "tail" beyond $(\nu+1)$st coordinate is innocuous from the general equilibrium point of view.

Note well that the new results, thus established for the limit hyperfinite economy, are more general in their own right than any predecessors by means of continuum approaches.

TABLE 2
HYPERFINITE LIMIT THEOREMS SUGGESTIVE OF ASYMPTOTIC AND ELEMENTARY THEOREMS

	Hyperfinite Limit Theorem with ${}^*\mathbf{R}^\nu$, $\nu \in {}^*\mathbf{N} - \mathbf{N}$	Hyperfinite Limit Theorem with ${}^*\mathbf{R}_\infty$	Asymptotic Theorem with $\mathbf{R}_\infty$	Elementary Theorem with $\mathbf{R}_\infty$
Equilibrium Existence	THEOREM 3 (NOMURA (1981)), Also THEOREM 1 in SECTION **2**)	THEOREM 2 (NOMURA (1981)), Also THEOREM 2 in SECTION **2**)	THEOREM 1 (NOMURA (1981)), Also THEOREM 3 in SECTION **2**)	THEOREM 1 (NOMURA (1993a)), Also THEOREM 4 in SECTION **2**)
Core Equivalence	Background Exercise for THEOREM 1 (NOMURA (1992c))	Background Exercise for THEOREM 1 (NOMURA (1992c))	Background Exercise for THEOREM 1 (NOMURA (1992c))	THEOREM 1 (NOMURA (1992c)), Also THEOREM 6 in SECTION **3**)

- **Of Lesser Impact:** Asymptotic interpretations of hyperfinite results that are derived in *Step 2* from the already known comparable results for the continuum counterpart, instead of being proven "from the scratch."

  Let $X \subset \mathbf{R}_+^\ell$ be the nonconvex commodity space, and the price p is chosen from $\{p \in \mathbf{R}_+^\ell \mid \|p\|_1 = 1\}$.

  It is well recognized that with an introduction of the nonconvexity of the commodity space, the *Dispersion Hypothesis* (*D.H.*) of sorts will suffice to restore the *upper hemicontinuity* of the individual demands, and consequently of the mean (excess) demand by reducing those making their choices at the "corner points" *negligible* economy-wide. (*D.H.*) may well take the following three specific guises in the respective economy:

– (*D.H.*) for the hyperfinite limit economy $\mathcal{E}_\omega : (a, \mathcal{I}(A), \lambda) \to {}^*\mathcal{P} \times {}^*X$: *For all* $p \in {}^*\Delta$, *and for all* $\alpha \in {}^*\mathbf{R}_+$, $\lambda\left(\{a \in A |\, p.e(a) = \alpha\}\right) \simeq 0$.

– (*D.H.*) for the purely competitive sequence of finite economies $\mathcal{G} = \{\mathcal{E}_n\}$: *For all* $\mathcal{E}_n \in \mathcal{G}$, *for all* $p \in \Delta$, *and for all* $\alpha \in \mathbf{R}_+$, *there exists an* $M \in \mathbf{N}$ *such that* $|\{a \in A_n |\, p.e(a) = \alpha\}| \leq M$.

– (*D.H.*) for the measurable limit economy $\mathcal{E} : (a, \mathcal{A}, \nu) \to \mathcal{P} \times X$: *For all* $p \in \Delta$, *and for all* $\alpha \in \mathbf{R}_+$, $\mu_p e(\alpha) = \nu\left(\{a \in A |\, p.e(a) = \alpha\}\right) = 0$.

TABLE 3

Hyperfinite Limit Theorems Derived from the Known Measurable Counterparts

	Measurable Limit Theorem for $\mathcal{E}$	Hyperfinite Limit Theorem for $\mathcal{E}_\omega$	Asymptotic Theorem for $\mathcal{G} = \{\mathcal{E}_n\}$
Equilibrium Existence	Theorem (Yamazaki (1978a))	Theorem 11 (Nomura (1984b))	Theorem 1 (Nomura (1984b))
Core Equivalence	Theorem 1 (Yamazaki (1978b))	Theorem 12 (Nomura (1984b))	Theorem 2 (Nomura (1984b))

Hyperfinite theorems are derived from the comparable measurable theorems

Nonstandard Analysis might well be evaluated as providing no more than an appropriate and straightforward apparatus for asymptotic interpretations when comparable limit results are already available for measurable economies, when compared with the alternative measure-theoretic approach pursued in Hildenbrand, Schmeidler and Zamir (1973).

- **Somewhat Successful:** In **5.1**, I demonstrated that the price characterization $\mathcal{V}_p(\mathcal{E})$ in place of the original $\mathcal{V}(\mathcal{E})$ helped to eliminate the convexity requirement on preferences, and thus generalized Aumann's Value Equivalence. Here, we shall sketch yet another generalization of admitting the nonconvexity of the commodity space.

We start by recalling the expenditure function $M(\succeq_a, p, x)$ introduced in **5.1**. As noted in **5.1**, $M(\succeq_a, p, x) = \min\{p.x' | \, x' \succeq_a x\}$ serves as a money metric representing $\succeq_a$. At a half way point in **5.1** in an attempt to get rid of convexity assumptions, we noted that a resort to the expenditure function enabled one to generalize Aumann's results to nonconvex preferences. Here, we go a step further, and investigate consequences of dropping the convexity of the commodity space as well.

With an introduction of the nonconvex commodity space, by now familiar (*D.H.*) needs be adapted, and is expressed in terms of the expenditure function distribution.

- (*D.H.*) for the hyperfinite limit economy $\mathcal{E}_\omega : (a, \mathcal{I}(A), \lambda) \to {}^*\mathcal{P} \times {}^*X$: *For all* $(p, x) \in {}^*\Delta \times \mathrm{con}\, {}^*\mathbf{X}$, *and for all* $\alpha \in {}^*\mathbf{R}_+$, $\lambda(\{a \in A | \, M(\succeq_a, p, x) = \alpha\}) \simeq 0$.

- (*D.H.*) for the purely competitive sequence of finite economies $\mathcal{G} = \{\mathcal{E}_n\}$: *For all* $\mathcal{E}_n \in \mathcal{G}$, *for all* $(p, x) \in \Delta \times \mathrm{con}\, X$, *and for all* $\alpha \in \mathbf{R}_+$, *there exists an* $M \in \mathbf{N}$ *such that* $|\{a \in A_n | \, M(\succeq_a, p, x) = \alpha\}| \leq M$.

- (*D.H.*) for the measurable limit economy $\mathcal{E} : (a, \mathcal{A}, \nu) \to \mathcal{P} \times X$: *For all* $(p, x) \in \Delta \times \mathrm{con}\, X$, *and for all* $\alpha \in \mathbf{R}_+$, $\mu_M(\alpha) = \nu(\{a \in A | \, M(\succeq_a, p, x) = \alpha\}) = 0$.

TABLE 4

HYPERFINITE LIMIT THEOREMS DERIVED FROM THE NEWLY ESTABLISHED MEASURABLE COUNTERPARTS

	Measurable Limit Theorem for $\mathcal{V}(\mathcal{E})$ and convex X	Measurable Limit Theorem for $\mathcal{V}_p(\mathcal{E})$ and nonconvex X	Hyperfinite Limit Theorem for $\mathcal{V}_p(\mathcal{E}_\omega)$ and nonconvex X	Asymptotic Theorem for $\mathcal{V}_p(\mathcal{E}_n)$ and nonconvex X
Value Existence	MAIN THEOREM (AUMANN (1964)), and THEOREM 1 (AUMANN (1975))	THEOREM 1 (NOMURA (1992b))	THEOREM 5.1 (NOMURA (1992b))	THEOREM 2 (NOMURA (1992b))
Value Equivalence	THEOREM 1 (AUMANN (1975))	THEOREM 3 (NOMURA (1992b))	THEOREM 5.2 (NOMURA (1992b))	Immediate. (Deduce from THEOREM 5.2.)

"Somewhat successful" in the following specific senses

- With the hyperfinite limit theorem as the ultimate goal, and subsequent asymptotic interpretation thereof in mind, we have managed to prove a standard measurable limit theorem by ourselves.
- The price characterization $\mathcal{V}_p(\mathcal{E})$ is reminiscent of utility assignment, and NEGISHI's taking advantage thereof in the seminal work (1969) with infinitely many commodities. It should not be difficult to generalize the present analysis to admit infinitely many commodities.

5.3.2 Elementary Theorems

As we demonstrated in SECTION **2**, it is potentially always possible to prove standard limit theorems corresponding to any hyperfinite limit theorems if one manages to develop standard proofs of the steps in the original nonstandard proofs that involve *external* entities. "Countability" is one such external entity, which is central in the "continuum" idealization of perfectly competitive markets with infinitely many agents by way of "countably" additive measure spaces, and the specification, by way of the space of sequences, of the space of infinitely many commodities contingent on "countably" many characteristics.

We are not as successful as we like in an attempt to maneuver elementary proofs by mimicking the original hyperfinite proofs and avoiding the external notions therein. The following TABLE 5 shows the state of the art of the elementary general equilibrium analysis, and assembles what few successful results we have accomplished. .

TABLE 5

ELEMENTARY THEOREMS WITH NONCONVEX PREFERENCES AND/OR COMMODITY SPACE

<table>
<tr><th></th><th>ELEMENTARY (Nonconvex $\mathcal{P}$ and Convex $X \subset \mathbf{R}^k_+$)</th><th>ELEMENTARY (Nonconvex $\mathcal{P}$ and Nonconvex $X \subset \mathbf{R}^k_+$)</th><th>ELEMENTARY (Nonconvex $\mathcal{P}$ and Convex $X \subset \mathbf{R}^+_\infty$)</th><th>ELEMENTARY (Nonconvex $\mathcal{P}$ and Nonconvex $X \subset \mathbf{R}^+_\infty$)</th></tr>
<tr><td rowspan="2">Equilibrium Existence</td><td rowspan="2">THEOREM (ANDERSON, KHAN and RASHID (1982))</td><td>Finite Dimensional Special Case of THEOREM 1 (NOMURA (1993a))</td><td rowspan="2">Special Case of THEOREM 1 (NOMURA (1993a))</td><td rowspan="2">THEOREM 1 (NOMURA (1993a), Also THEOREM 4 in SECTION 2)</td></tr>
<tr><td>HYPERFINITE (Nonconvex ${}^*\mathcal{P}$ and Nonconvex ${}^*X \subset {}^*\mathbf{R}^k_+$) THEOREM 6 (KHAN and RASHID (1982))</td></tr>
<tr><td>Core Equivalence</td><td>THEOREM 1 (ANDERSON (1978))</td><td>Finite Dimensional Special Case of THEOREM 1 (NOMURA (1992c))</td><td>Special Case of THEOREM 1 (NOMURA (1992c))</td><td>THEOREM 1 (NOMURA (1992c), Also THEOREM 4 in SECTION 3)</td></tr>
</table>

5.3.3 Loeb Measure

It used to be the case that, due to the *external* nature in the hyperfinite context of the continuum as idealized by countably additive measure space of agents, the hyperfinite results did not admit directly comparable implications for continuum economies. The breakthrough was provided by the construction of measure spaces due to LOEB (1975), known as Loeb measures.

In what follows, we shall summarize only those properties that we have actually taken advantage of in the course of our past applications (see NOMURA (1992)). For more complete account, we refer to ANDERSON (1976), (1982) and (1991), and LOEB (1979). In particular, ANDERSON (1982) shoed that Loeb's construction enables one to obtain a large class of measure spaces from the hyperfinite models, including Radon spaces, and also extended applications to Baire spaces in the case of non-Radon spaces.

Loeb's construction starts with $(X, \mathcal{T}, \nu)$, an internal measure space in a denumerably comprehensive enlargement of a superstructure containing

$\mathbf{R}$, where X is an internal set in this enlargement, $\mathcal{T}$ an internal algebra of subsets of X, and $\nu : \mathcal{T} \to {}^*\mathbf{R}_+$ an internal finitely additive measure.

Loeb shows that the set X, considered now as a standard set, is a standard measure space when equipped with the smallest σ-algebra $\sigma(\mathcal{T})$ containing $\mathcal{T}$, and with the standard countably additive measures $L(\nu)$ on $\sigma(\mathcal{T})$ obtained as the unique extension of standard part of ν to $\sigma(\mathcal{T})$ by the Caratheodory procedure.

Proposition 7 (Loeb (1975, Thorem 1)): *Let $(X, \mathcal{T}, \nu)$ be an internal measure space in a denumerably comprehensive enlargement. Then, there is a unique countably additive standard measure $L(\nu)$ on $\sigma(\mathcal{T})$, the smallest σ-algebra in X containing $\mathcal{T}$, such that for each internal set $A \subseteq X$, $L(\nu) = \mathrm{st}(\nu(A))$.*

The following observation is due to Rashid (1979, Lemma 1).

Proposition 8 (Rashid (1979, Lemma 1)): *If ν is an infinitesimal measure, i.e., $\nu(x) \simeq 0$ for all $x \in X$, then $L(\nu)$ is non-atomic.*

Loeb further shows that internal, $\mathcal{T}$-measurable functions are converted to extended real-valued $\sigma(\mathcal{T}$-measurable function on X.

Proposition 9 (Loeb (1975, Thorem 2)): *Let $f : X \to {}^*\mathbf{R}$ be an internal, $\mathcal{T}$-measurable function. Then, $g : X \to \mathbf{R} \cup \{+\infty, -\infty\}$ given by $g(x) = \mathrm{st}(f(x))$ is $\sigma(\mathcal{T})$-measurable.*

$f : X \to \ * \mathbf{R}$ is said to be *S-integrable* if

(i) f is $\mathcal{T}$-measurable,

(ii) $\mathrm{st}\left(\sum_{x \in X} |f(x)|\nu(x)\right) < +\infty$,

(iii) $A \subseteq X, \nu(A) \simeq 0 \Rightarrow \sum_{x \in A} |f(x)|\nu(x) \simeq 0.$

Anderson (1976, Theorem 6) generalized to $S-$integrable functions Loeb's integrability result (1975, Theorem 3) for finite functions. Note that if f is finite, then $\mathcal{T}$-measurability implies S-integrability.

Proposition 10 (Anderson (1975, Theorem 6)): *Let $f : X \to {}^*\mathbf{R}$ be S-integrable, and let $g(x) = \mathrm{st}(f)$. Then, g is $L(\nu)$-integrable and*

$$\int_A g(x) dL(\nu)(x) = \mathrm{st}\left(\sum_A f(x)\nu(x)\right)$$

for each $A \subseteq X$.

The following result on the recoverability of $L(\nu)$-integrable functions as S-integrable functions on $(X, \mathcal{T}, \nu)$ is due to ANDERSON (1976, THEOREM 7), which again generalizes LOEB's comparable result for bounded $\sigma(\mathcal{T})$-measurable functions (1975, PROPOSITION 2).

PROPOSITION 11 (ANDERSON (1975, THEOREM 7)): *Suppose* $g : X \to \mathbf{R}$ *is* $L(\nu)$*-measurable. Then, there is an* S*-integrable* $f : X \to {}^*\mathbf{R}$ *such that* $\mathrm{st}\,(f(x)) = g(x)$ $L(\nu)$*-almost everywhere.*

5.3.4 "Market Thickness," or the Relative Size Requirement

- The following quotation from MAS-COLELL (1975, p.265) best summarizes the judicious concern that the infinite dimensional commodity spaces open up a possibility of infinite variations in agents' characteristics, and the consequent need to resort to a version of the remedial *relative size* requirement, or the *market thickness* requirement according to GRETSKY and OSTROY (1985) coinage, in order to secure the *negligibility* of agents in *each* of infinitely many markets:

  > ... , although infinite-dimensional, the 'size' of the commodity space is sufficiently small relative to the size of the economy for the equality of core and equilibria to obtain [this is a well-known heuristic requirement for Aumann's theorem to be generalizable; ...]; ... .

  - With the infinite-dimensional commodity space, it is *market thickness*, not *nonatomicity* of the space of agents that generates such economic consequences as existence and core-equivalence of competitive equilibria. Incidentally, an economy with a nonatomic space of agents and finitely many commodities is always thick.
  - With a Banach lattice as the commodity space and a nonatomic space of agents, GRETSKY and OSTROY (1985) show that if an allocation vector measure is representable as an integral with respect to the nonatomic measure on the space of agents, i.e., has Radon-Nikodym derivative, then markets are thick.

- OSTROY (1984, 143-144) notes:

> There is a tension between these two infinities [the number of commodities and that of agents]. It is known that the competitive properties of Walrasian equilibrium in models with a continuum of agents requires some restrictions on the commodity space so as to preclude 'truly large-square' models where 'the number of commodities is as large as the number of agents'. In these truly large-square models, the potential competition from large numbers of agents is offset by the variety of commodities in the sense that infinitesimal agents may trade commodities for which there are no good substitutes

Despite many similarities between measure spaces and hyperfinite sets, idealizing the limit of the sequence of spaces of the finitely many agents (indeed, the Loeb measure construction converts a hyperfinite set to a measure space), Anderson (1991, p.2148) calls attention to some explicated assumptions in the hyperfinite context, hidden in the measure theoretic formulation:

> However, there are certain phenomena that can occur in hyperfinite economies which are ruled out by the measure-theoretic formulation. ... In the hyperfinite context, certain conditions inherent in the measure-theoretic formulation can be seen to be strong endogenous assumptions. Using hyperfinite exchange economies, we can state exogenous assumptions which imply the endogenous assumptions inherent in the measure-theoretic formulation, as well as explore the behavior of economies in which the endogenous assumptions fail.

As such, the "Market Thickness" condition becomes explicit in the hyperfinite idealization.

(γ.1) *Hyperfinite Relative Size Requirement* (Nomura (1981, Theorem 3, Assumption (*iv*); Also reproduced in the present article as Theorem 1, Assumption (c))[7]): Let $\nu \in {}^*\mathbf{N} - \mathbf{N}$ be the number of commodities and $\omega \in {}^*\mathbf{N} - \mathbf{N}$ the number of agents, where ${}^*\mathbf{N}$ is the nonstandard extension of $\mathbf{N}$. Then,

$$\frac{\nu}{\sqrt{\omega}} \simeq 0.$$

[7] Lewis (1977, Essay I, Theorem 1) and Brown and Lewis (1981, Theorem II.1, Assumption (*iv*)) postulate a comparable assumption which claims $\dfrac{\nu}{\omega} \simeq 0$ instead.

In retrospect, relative size requirements, in several guises, might well be understood to have served, in not necessarily mutually exclusive ways, to secure for subsets of infinite dimensional commodity spaces the following three desired properties in equilibrium existence analysis: (α) *compactness*, in the topology with respect to which the preferences are continuous, (β) *non-empty interior*, and (γ) *average convexity*. Some detailed accounts are in order:

> **(α) Compactness:** Except for the straightforward compactness of the cartesian product of any nonempty collection of compact spaces in the product topology, as guaranteed by Tychonoff's Theorem (see e.g. DUNFORD and SCHWARTS (1958, Section I.8.1)), the difficulty (α) stems from the fact that, unlike the finite dimensional case, closed and bounded subsets of infinite dimensional spaces need not be compact.
>
> > (α.1.1) *Impatience, or Myopic Preferences* (BROWN and LEWIS (1981), and RAUT (1986)): Any continuous preferences on ℓ_∞ (BROWN and LEWIS (1981)) or on L_∞ (RAUT (1986)), in the locally convex Hausdorff topologies coarser than the Mackey topology, exhibit *strong myopia* in the sense that finite differences in sufficiently distant *tails* make no change in choices made in the light of continuous preferences in such topologies.
> >
> > (α.1.2) *Topologically Separable Commodity Space* (GABSEWICZ (1968 and 1991)): The commodity space is restricted to $C(S)$, the set of bounded continuous functions on a compact metric space S, that is topologically separable, which enables one to restrict the size of the commodity space.
> >
> > (α.1.3) *Measurability of Agents' Characteristics* (BEWLEY (1973, p.386)): The preference relation of an agent is Lusin measurable in the Hausdorff uniformity derived from the uniformity of the Mackey topology on the commodity space L_∞; his initial endowment is measurable in the sup norm; and the production set is measurable in pseudo-metric, thus reducing agents' diversity.
> >
> > (α.1.3') *Strong Measurability of Agents' Characteristics* (MERTENS (1970 and 1991)): The commodity space is chosen to be L_∞ on (S, Σ, σ), a totally σ-finite *separable* positive measure space, in addition to the measurability of agents' preferences and initial endowment.

(α.2) *Compactness of the Space of Commodity Characteristics* (Mas-Colell (1975, (I))): The space of commodity characteristics K is chosen to be a compact metric space, and the commodity space is restricted to $C(K)$, the set of bounded continuous functions on K. The commodity space is further restricted to a discrete subspace with one perfectly divisible commodity, where "the margin of choice is in the commodity chosen rather than in the quantity purchased of a given commodity (Jones (1984, p.514))." When combined with the compactness of agents' characteristics (Mas-Colell (1975, (VIII))), the compactness requirement of K serves to restrict the size of the commodity space sufficiently small relative the the size of the economy.

(α.3) *"Market Thickness" as a Dimensional Restriction on the Final Allocation* (Gretsky and Ostroy (1985)): An exchange economy is said to exhibit *thick markets* if the final allocation mapping is *Dunford-Pettis*. If the commodity space, the range of the allocation mapping, is endowed with the norm topology, and if it is Bochner integrable, which is tne appropriate choice of the representability for the norm topology, then the allocation mapping is Dunford-Pettis. The market economy is thick if and only if the allocation mappings have range with compact and convex closures (Gretsky and Ostroy (1985, Section 3.1, Theorem and Remark 1)).

(α.4) *"Physical Thickness" as a Dimensional Restriction on the Endowment of the Agents* (Ostroy and Zame (1994) and Anderson (1990)): Markets are said to be *physically thick* if the endowment map $e : A \to \mathcal{M}_+(T)$ satisfies: there is a constant K such that $e(a) \leq Km$ for almost every $a \in A$, where $m \in \mathcal{M}_+(T)$ is a *reference bundle* such that m is nonatomic and $\sup m = T$ (Ostroy and Zame (1994, Definition)). If e is weak* measurable, and $e(a) \leq Km$ for all $a \in A$, then e is norm measurable and hence Bochner integrable (Ostroy and Zame (1994, Footnote 4)). Therefore, e is almost a sure limit of functions taking on only a finite number of values.

Together with an assumption on the bounded rates of substitution (Ostroy and Zame (1994, (B))):

> there is a constant M such that if $X, Y, Z \in \mathcal{M}_+(T)$, and if $Z - X + Y \geq 0$ and $M \parallel X \parallel \leq \parallel Y \parallel$, then $Z - X + Y \succ_a Z$ for all $a \in A$,

that is much weaker than the *economic thickness* condition (β.1), the physical thickness condition serves to restricting the range of final allocations dimensionally to an order interval in the commodity space (OSTROY and ZAME (1994, THEOREM 4)):

ANDERSON (1990, SECTION 4) is well aware of the necessity of some kind of relative size requirement, and goes on to propose for economies with a large but finite number of agent the following analogue of OSTROY and ZAME's physical thickness condition:

> The dimension of the linear span of the set of individual endowments is much smaller than the number of agents.

(β) Non-empty Interior: *Uniformly proper* preferences ensure non-empty interior of the supported set, and consequently the existence of supporting prices.

Initially little attention was paid to (β), because BEWLEY's (1972) choice of the commodity spaces, the positive cone of ℓ_+ or L_+, has non-empty interior. Indeed, as MAS-COLELL (1986, p.1043) points out, for strictly positive points of L_∞ with the Mackey topology, the uniform properness is implied by monotonicity, convexity and continuity of preferences. As noted in the preceeding discussion of (α.2), despite the empty interior of the positive cone of $C(K)$, the space if continuous functions on K, MAS-COLELL (1975) managed to get around the empty interior problem by restricting to the discrete subset of $C(K)$.

(β.1) *Uniform Substitutability* (JONES (1983) and JONES (1984)) or *Economic Thickness* (OSTROY and ZAME (1994), and its *Asymptotic Version* (ANDERSON (1991, SECTION 3)):

With $\mathcal{M}(T)$, the finite nonnegative signed measures on the space of commodity characteristics T, equipped with the distance d, as the commodity space, JONES (1983, (UHS)) and JONES (1984, *Assumptions* HS1-HS3) restrict the preferences to those exhibiting the *Uniform Substitutability* Property. Markets are said to be "*economically thick*" if the preferences therein satisfy the *Uniform Substitutability* Property:

> For all $\gamma > 1$, there exists $\rho > 0$ such that for all $\alpha > 0$, for all $m \in \mathcal{M}(T)$, and for all $s, t \in T$ with $d(s,t) < \rho$,
>
> $$m + \alpha\gamma\delta_t \succ m + \alpha\delta_s,$$
>
> where δ_t is the Dirac measure at t, i.e., $\delta_t(S) = 1$ if $t \in S \subset T$ and 0 otherwise.

That is, for every given favorable *terms* $\gamma > 1$, there exists an upper bound ρ on the similarity of characteristics so that, for trades involving commodities with the specified similar characteristics, any trade in which the terms are favorable is preferred. Intuitively, sufficiently close characteristics, $s, t \in T$ are good substitutes at the margin, in the sense that their marginal rate of substitution is uniformly close to unity. The preference relation which has a utility representation in such a way that marginal utility depends continuously on the commodity characteristics, is said to be *proper* (Jones (1984, Proposition 1)). Jones (1984, p.514) goes on to claim: " ...The simultaneous choice over both of these margins can be handled as long as the quantity margin is more important than the characteristics one." Thus, the sufficient substitutability between the consumption characteristics ensures the validity of the approximation of the infinite dimensional commodity space by large but finite dimensional commodity spaces.

Anderson (1990, Section 3) gives an *Asymptotic Characterization of Uniform Substitutability* on $\mathcal{M}([0,1])_+$, the space of countably additive finite nonnegative Borel measures on $[0,1]$:

> Partition $[0,1]$ into k intervals
> $$\left[0, \frac{1}{k}\right), \left[\frac{1}{k}, \frac{2}{k}\right), \ \ldots \ , \left[1 - \frac{1}{k}, 1\right],$$

and

> Identify a finite set
> $$\left\{ t_i \middle| \, t_i \in \left[\frac{i-1}{k}, \frac{i}{k}\right) (i = 1, 2, \ \ldots \ , k-1); t_k \in \left[1 - \frac{1}{k}, 1\right] \right\}$$
> with the property that any $t \in [0,1]$ is nearly a perfect substitute for some t_i.

(β.2) *Uniformly Proper Preferences* (Mas-Colell (1986)): The commodity space L_+ is chosen to be the positive cone of a locally convex, Hausdorff ordered vector space L. In the vein of imposing bounds on the marginal rate of substitution, Mas-Colell (1986, Definition, p.1043) restricts the preference relation $\succeq$ to a *uniformly proper* class:

> At every $x \in L_+$, there exist $\nu \geq 0$ and an open neighborhood of the origin V, both of which can be chosen independently of x, such that
> $$[z \in L] \wedge [x - \alpha\nu + z \succeq x] \Rightarrow [z \in \epsilon V].$$

In words, the marginal rates of substitution for the given commodity bundle ν are assumed to be bounded in the sense that, if is is impossible to compensate for a loss of $\alpha\nu$ with z, then z is too small relative to $\alpha\nu$.

Richard and Zame (1989, Theorems 2 and 4) establish the essential equivalence of (β.2) to the extensibility of $\succeq$ on L_+ to that on convex subset of L containing L_+ with nonempty interior.

(β.3) *Existence of Extremely Desirable Commodities* (Yannelis and Zame (1986)) as a non-transitive variant of (β.2) : For $x \in L_+$ and $\nu \in L_+$, define

$$\Gamma_\nu(x) = \left\{ \mu \in \mathbf{R}_+ \middle| \begin{array}{c} x - \alpha\nu + z \succ x;\ -1 \leq \alpha < 0; \\ x - \alpha\nu + z \in L_+;\ \|x\| < -\alpha\mu \end{array} \right\}.$$

There exists an *extremely desirable* commodity bundle ν in the sense that

$$\inf \{\mu(\nu, x) |\, x \in L_+\} > 0,$$

where $\mu(\nu, x) = \max\{\mu |\, \mu \in \Gamma_\nu(x)\}$ measures the marginal rate of substitution of ν for x. That is, ν is said to be *extremely desirable* if it is possible to compensate the loss of z, to the extent of $\|z\|$, with the increment of $-\alpha\nu$ so long as $\|z\|$ is sufficiently small relative to α.

It is not difficult to see that, for transitive, complete, convex and non-interdependent preferences, (β.2) and (β.3) are equivalent.

(γ) Average Convexity: Not surprisingly, (γ) has received yet less attention, since the existence and/or core-equivalence proofs proceed by securing the extesibility of the comparable results with possibly approximations for the economies with finitely truncated commodity spaces. As far as the finite truncations are concerned, the property (γ) holds automatically for each constituent truncated economy without any further assumptions.

However, one should be aware that the convexity of the preferences and that of the commodity space need to be retained in order to secure the desired extensibility, either by taking the limit of finite dimensional economies, or by converting infinite dimensional allocations to the corresponding finite dimensional utility assignments to finitely many agents (except possibly for Yannelis and Zame (1986) that does not assume the convexity of preferences, and Mas-Colell (1975) with the nonconvex commodity space due to commodity indivisibility). Two notable exceptions to such state of the art are (γ.1) and (γ.2).

(γ.1): Already stated in a "box" at the onset of the present discussion of the "Market Thickness."

(γ.2) *Finitely Spannable Commodity Space* (NOMURA (1993, THEOREM 1, *Assumption* 1; Also reproduced in the present article as THEOREM 4, *Assumption* 1): For all $a \in A$, $d_a(p)$ has a finite family of convex subsets $\left\{d_a^j(p) \mid j = 1, \ldots, \kappa_a\right\}$ such that

$$d_a(p) = \bigcup_{j=1}^{\kappa_a} d_a^j(p).$$

In spirit, (γ.2) has the closest bearings on the characterization (α.3) of the physical thickness due to GRETSKY and OSTROY (1985), in that both essentially restrict the dimensions of the final allocations.

Mathematically, it is worth pointing out that an application of either of two major Average Convexity Theorems is problematic in the presence of infinitely many commodities. LYAPUNOV THEOREM, applicable to economies with a measure space of agents, fails in every infinite dimensional Banach space (see e.g. DIESTEL and UHL (1977, SECTION IX.1)). Alternatively, the degree of approximation explicated by an application of SHAPLEY-FOLKMAN THEOREM, due originally to STARR (1969, APPENDIX), for economies with a finite number of agents, depends crucially on the finite dimensionality of the commodity space, and therefore inapplicable.

5.3.5 More on Finite Spannability

Needs for (γ.2) *Finite Spannability* in **5.3.4** were never recognized in the finite dimensional context where (γ.2) always follows by CARATHEODORY'S THEOREM (see e.g. HILDENBRAND (1974, p.37)), except possibly for BROOM (1972, *Assumption* 2.6), that required finite spannability by non-corner points.

A Measure-Theoretic Analogue of Finite Spannability: In order to argue convincingly in support of somewhat artificial assumption (γ.2) *Finite Spannability*, we draw our attention to measure-theoretic analogues of average convexity theorems, known as the Lyapunov Theorems, and in particular the following extension to the infinite dimensional spaces due to Akemann and Anderson (1991), where needs for similar representability conditions were recognized.

The measure-theoretic Lyapunov's Theorem (see e.g., Hildenbrand (1974, p.45)), of which Shapley-Folkman Theorem constitutes the finite analogue, establishes that the range of a finite dimensional vector-valued measure is compact and convex. When restated in the language of operator algebras, Lyapunov's Theorem reads as:

Proposition 12 (Lyapunov Theorem for Nonatomic von Neumann Algebras (Akemann and Anderson (1991, Theorem 2.5)): *If Ψ is a weak* continuous linear map from an abelian, nonatomic von Neumann algebra $\mathcal{M}$ to a finite dimensional space, then $\Psi(P) = \Psi((\mathcal{M}_+)_1)$, where P denotes the set of projections, i.e., extreme points, in $\mathcal{M}$ and $(\mathcal{M}_+)_\infty$ denotes the positive portion of the unit ball of $\mathcal{M}$.*

In hindsight, the proof of Proposition 12 hinges not surprisingly on the dimension of the domain being strictly greater than that of range, the latter of which is taken to be finite.

In order to generalize Proposition 12 to Ψ with the infinite dimensional range, and to comprehend the restrictive meanings of the additional assumptions therein, we need to assemble some definitions from operator algebras.

> A continuous linear functional f on a von Neumann algebra $\mathcal{M}$ is *singular* if and only if for each nonzero projection p in $\mathcal{M}$, there is a nonzero projection $q \leq p$ such that $f(q) = 0$. Also, a bounded linear map Ψ from a von Neumann algebra$\mathcal{M}$ into a normed linear space $\mathcal{X}$ is said to be *singular* if $\Psi^*(\mathcal{X}^*)$ consists entirely of singular functional on $\mathcal{M}$ (Akemann and Anderson (1991, p.50)).

> The following definitions weaken the notion of separability: A von Neumann algebra $\mathcal{M}$ is *essentially countably decomposable* if, given a singular state f on $\mathcal{M}$, and a family $\{p_\alpha | \, \alpha \in \kappa\}$ of orthogonal projections in the *kernel* of f, $\mathrm{Ker}(f) = \{f(p) = 0\}$, i.e., $f(p) = 0$ for each α, with $\sum_{a \in \kappa} p_\alpha = 1$ (which is always possible by Zorn's Lemma (see e.g. Dunford and Schwartz (1968, Theorem I.2.7)), there is a partition of κ into a countable family of disjoint subsets $\{\kappa_n\}$ such that

if we write $p_n = \sum_{\alpha \in \kappa_n} p_\alpha$, then each p_n lies in Ker(f) (AKEMANN and ANDERSON (1991, DEFINITION 6.1)).

A von Neumann algebra $\mathcal{M}$ is *κ-decomposable* if every family of orthogonal projections in $\mathcal{M}$ has cardinality less than κ, the smallest cardinal with this property, and is said to be *not too large* if it is κ-decomposable for some submeasurable cardinal κ (AKEMANN and ANDERSON (1991, p.54)).

The following result due to AKEMANN and ANDERSON (1991, PROPOSITION 6.4) establishes that a von Neumann algebra $\mathcal{M}$ is *essentially countably decomposable* if either of the following conditions holds:

(1) $\mathcal{M}$ is countably decomposable;

or

(2) $\mathcal{M}$ is not too large, and the continuum hypothesis holds true.

Let z_a, the supremum of all the minimal projections in $\mathcal{M}$, be a central projection in $\mathcal{M}$ (which does exist since the family of minimal projections in $\mathcal{M}$ is unitarily invariant (AKEMANN and ANDERSON (1991, p.2)). Write the atomic algebra $\mathcal{M}_a = z_a\mathcal{M}$ and the finite part of $\mathcal{M}_a$, $\mathcal{M}_{\text{fin}} = z_{\text{fin}}\mathcal{M}$, where $z_{\text{fin}} = \sup\{z \in \mathcal{M}_a |\, z$ is a central partition and $z\mathcal{M}_a$ is finite dimensional$\}$.

With these definitions in hand, we are now in the position of presenting an infinite dimensional generalization of the preceding PROPOSITION 12, which may be reckoned as a continuum analogue of our AVERAGE CONVEXITY THEOREM OF INFINITE DIMENSIONAL RANGES (NOMURA (1993a), reproduced as THEOREM 8 in SECTION 4) *a la* SHAPLEY-FOLKMAN THEOREM.

PROPOSITION 13 (LYAPUNOV THEOREM for Singular Maps (AKEMANN and ANDERSON (1991, THEOREM 6.12)): *If $\mathcal{M}$ is an essentially countably decomposable von Neumann algebra such that the center of the finite part of $\mathcal{M}$ is finite dimensional, if $\mathcal{X}$ is a normed linear space where the dual space $\mathcal{X}^*$ is weak* separable, and if Ψ is a singular map of $\mathcal{M}$ into $\mathcal{X}$, then $\Psi(P) = \Psi((\mathcal{M}_+)_\infty)$, where P denotes the set of projections, i.e., extreme points in $\mathcal{M}$, and $(\mathcal{M}_+)_\infty$ denotes the positive portion of the unit ball of $\mathcal{M}$.*

To sum up, (γ.2) *Finite Spannability* is no more restrictive than postulating the weak* separability of $\mathcal{X}^*$, together with the essentially countable decomposability of $\mathcal{M}$, in the measure-theoretic counterpart. Both may well be looked upon as the *relative size requirement*. Intuitively, in our SHAPLEY-FOLKMAN-TYPE AVERAGE CONVEXITY THEOREM in which the domains are finite dimensional, the ranges need to be at least of finite structure. By comparison, since the domain of a singular map Ψ is of "essentially" inseparable nature, which is further restricted to the essentially countably decomposable class, it may be hoped that an infinite dimensional extension of Lyapunov Theorem, such as PROPOSITION 13 is only possible by securing some sort of separability for the range of singular maps, more specifically the weak* separability of the dual of the domain. It is also by now apparent that (i) of COROLLARY 4 in SECTION **4**, for the finitely decomposable domain, each corresponding to a type of consumers, has close bearings on the hypothesis in PROPOSITION 2.

Approximate Asset Price Equilibria: *Assumption* $(\gamma.2)$ has close connections with spannability of incomplete financial markets by finitely many marketed assets.

Let S be a separable metric space. $s \in S$ is a state of the world. Let (S, Σ, μ) be a probability space, with μ a finite measure on S , absolutely continuous w.r.t. the Lebesgue measure on S. $L_2(\mu)$ is a normed linear space of real-valued measurable functions f defined on S, of which the L_2-norm $\|f\|_2 = (f.f)^{\frac{1}{2}} = \left(\int_S f^2 d\mu\right)^{\frac{1}{2}}$ is finite. Then, L_2 is a separable Hilbert space, and emerges as a natural candidate for the commodity space of financial markets where contingent claims are typically of finite means and variance.

By the Cauchy-Schwartz Inequality, the space of square-integrable contingent claims is a Hilbert space. Thus,

$$X \subset L_2^+ = \{f \in L_2 |\, f \geq 0\}$$

denotes the space of contingent claims. L_2 will be endowed with the L_2-norm topology $\mathcal{T}_2$.

Furthermore, when S is decomposed as $S = \{s_1\} \cup S_d$ after appropriately reindexing the states of the world, and when the real-valued measurable function f defined on S takes the values, $f(s_1) \in \mathbf{R}_+$ and $f|_{S_d} \in \mathbf{N} \cup \{0\}$, define X_d as the subset of X consisting of such f's, i.e.,

$$X_d = \left\{f \in L_2^+ |\, f(s_1) \in \mathbf{R}_+, f|_{S_d} \in \mathbf{N} \cup \{0\}\right\} \subset X.$$

Then, X_d is the space of *discrete* square-integrable state-contingent claims, all of which except one are transacted discretely in integer values.

Marketed Securities. $M_0 \subset X$ with car$M_0 - \nu \in \mathbf{N}$ is the space of *marketed securities*, i.e., claims to state-contingent consumption at the terminal date, say T for which a market exists. These ν long-lived assets or securities allow agents to consumption across dates and states. For $x \in M_0$, x^i, the i-th coordinate of x, denotes the amount of the i-th security that entitles the bearer on date T to $\delta^i(s)$ units of the consumption at date T if the state is s. Thus, in the presence of ν marketed securities, the security markets are characterized by a dividend process δ, the valuation functional p_0 on M_0 and the associated gain process G expressed as $G(t) = p_0(t) - p_0(0) + \delta(t),\ t \in [0, T]$.

Consider options constructed solely of the marketed securities, and define the subspace $M \subset X$ by spanM_0, the span of M_0. The valuation functional p is the price of the options in M, which extends p_0 to M.

Information Revelation Process. Consider a filtration $\{\mathcal{F}_t\}$ towards the σ-algebra Σ on S. Let $P : \Sigma \to [0, 1]$ be the equivalent class of subjective probabilities held by agents. Given (S, Σ, P), the *filtration* is the information *commonly held* by agents at $t \in [0, 1]$ such that $\mathcal{F}_{t+1}$ is at least as fine as $\mathcal{F}_t$, $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \Sigma$[8]. At t, *all* agents know which cell of $\mathcal{F}_t$ contains the true state, and thus, $\{\mathcal{F}_t\}$ specifies the order in which uncertain events are revealed to be true or false over $t \in [0, T]$.

We may well limit the space of contingent commodities X to a single physically identifiable consumption good, available at any date $t \in [0, 1]$, and consumption of which is adapted to $\mathcal{F}_t$. Thus a generic element $x \in X$ is such that $x(t)$ is a $\mathcal{F}_t$-measurable function on S, the value of which is denoted by $x(t, s)$ in the state s.

Trading Strategy. Given the state space S endowed with the σ-algebra Σ and the filtration $\{\mathcal{F}_t\}$, define the *trading strategy* of agent a $\theta : S \times [0, T] \times A$ in terms of a portfolio of marketed securities. $\theta(s, t, a)$ is assumed to be $\mathcal{F}_t$-measurable, and satisfy $p_0\theta(s, t, a) \in L_2$.

Given p_0, the trading strategy $\theta(s, t, a)$ is converted to the following *net trades in state-contingent consumption*:

$$x(\theta, p_0, a) = (-\theta(0, a)p_0(0), \Delta\theta(t, a)p_0(t), \Delta\theta(T, a)p_0(T) + \theta(T, a)\delta)$$

REMARK 11: The original BLACK and SCHOLES' model (1973) assumes that agents have endowments and consume only at date 0 and T, and restricts

[8] At $t = 0$, no information is available to agents about the state of the world, while at $t = T$ agents learn all information to conclude the true state of the world.

the trading strategies to the self-financing class, i.e., $\Delta\theta(t)p_0(t) = 0$ for all $t \in [0, T]$.

Following KREPS (1982), we define by the number of *subcells* of F in $\mathcal{F}_{t+1}$ a measure of the amount of information that might be received by date $(t+1)$ if at date t the cell F is known to prevail, i.e.,

$$K(t, F) = \text{Card}\left(\{F' \in \mathcal{F}_{t+1} | \, F' \subseteq F\}\right).$$

In the present context of incomplete financial markets, (γ.2) *Finite Spannability* may well be expressed as:

Assumption 9 (Finite Spannability): There exists a large but finite number ν of marketed long-lived assets in M_0, defined by

$$\nu = \max\{K(t, F) | \, t < T, \; F \in \mathcal{F}_t\}.$$

REMARK 12: Define $M(a) \subset M$ by

$$M(a) = \max\{x \in X | \, (\exists r(a) \in \mathbf{R}) \, (\exists \theta(a)) \, x - r(a) \in x(\theta(a), p_0)\}$$

and $p : M(a) \to \mathbf{R}$ such that $p.\, (r(a) + x(\theta(a), p_0)) = r(a)$. Then, KREPS (1982, PROPOSITION 2) establishes that our *Assumption* 9 is a necessary and sufficient condition for $M(a) = X$.

By a slight abuse of notation, *Assumption* 2 (Bounded Nonconvexity) reads in the present context as:

Although we have not yet managed to delve into its behavioral interpretations in terms of agents' characteristics, the BOUNDED NONCONVEXITY ASSUMPTION is so far stated as postulating sufficient risk aversion so that the resulting demands are bounded.

Assumption 10 (Risk Aversion): There exists a scalar $\Lambda > 0$ such that $\max\{d(p, a) | \, a \in A\} \leq \Lambda$.

Resorts to *Assumption* 9 (Finite Spannability) and *Assumption* 10 (Sufficient Risk Aversion) establish an ELEMENTARY EQUILIBRIUM EXISTENCE THEOREM (NOMURA (1986; Revised, 1991, THEOREM 1), analogous to NOMURA (1993a, THEOREM 1; Also reproduced as THEOREM 4 in the preceding SECTION **2**).

With an introduction of discrete transactions, the *Dispersion Hypothesis* (*D.H.*) now reads as:

- $(D.H.)$: Given a real-valued function $\alpha : S_d \to \mathbf{R}_+$, there exists a scalar $\lambda \in \mathbf{R}_+$ such that

$$\sum_{s \in S_d} \mid \left\{ a \in A | e^1(a) = \alpha(s) \right\} \mid \leq \lambda . n.$$

The above $(D.H.)$ restores the desired upper hemicontinuity of the average demand, which in turn enables us to generalize the Elementary Proof to the nonconvex commodity space as well (NOMURA (1982, THEOREM 2), in a comparable manner to (NOMURA (1993a, THEOREM 1); Also reproduced as THEOREM 5 in SECTION **2**).

5.3.6 Mixed Markets with Atoms

Incorporations of mixed structures will be bidirectional, i.e., in the space of agents and/or in the space of commodities.

- **Spaces of Agents:** Consisting of an ocean of small agents plus syndicates.

  - *Measure-Theoretic Characterization of Mixed Space of Agents* (SHITOVITZ (1973)):

  - *Hyperfinite Characterization of Mixed Space of Agents* (KHAN (1976, SECTION **5**)): Let A_ω with $|A_\omega| = \omega \in {}^*\mathbf{N} - \mathbf{N}$ be divided into two groups, $A_1 = \{a_1, a_2, \ldots, a_m\}$ and $A_0 = \{a_{m+1}, a_{m+2}, \ldots, a_\omega\}$ for some $m \in \mathbf{N}$. Each agent carries his *internal weight* $\lambda(a) \gtrsim 0$ such that $\sum_{a \in A_\omega} \lambda(a) = 1$.

    Agents in A_1 will be identified as "large" agents, while those in A_0 as "small agents" in the following sense (KHAN (1976, SECTION **5**, A.6, p.287)):

    $$(\forall a \in A_1)[\lambda(a) \gtrsim\!\!\!\!/\; 0],$$

    and

    $$(\forall a \in A_0)[\lambda(A) \simeq 0].$$

- *Finite Characterization of Mixed Space of Agents* (Adapted from Khan (1976, Section **2**)): Let A with $|A| = n \in \mathbf{N}$ be divided into two groups, $A_1 = \{a_1, a_2, \ldots, a_m\}$ and $A_0 = \{a_{m+1}, a_{m+2}, \ldots, a_n\}$ for some $m < n$. Each agent carries his *weight* $\lambda(a) \geq 0$ such that $\sum_{a \in A} \lambda(a) = 1$.

  Agents in A_1 will be identified as "large" agents, while those in A_0 as "small agents" in the following specific sense (Khan (1976, Section **2**, (4), p.279)):

  $$(\exists \xi \in \mathbf{R}_+)(\forall a \in A_1)[\lambda(a) \geq \xi],$$

  and

  $$(\forall \delta \in \mathbf{R}_+)(\exists \nu \in \mathbf{N})[|T_0| > \nu \Longrightarrow (\forall a \in A_0)\lambda(A) \leq \delta].$$

- **Spaces of Commodity Characteristics:** Consisting of closely substitutable characteristics plus distinct characteristics or possibly complementary characteristics jointly consumed, that will warrant the emergence of the market power exploited by a limited number of "large" agents.

  - *Measure-Theoretic Characterization of the Commodity Space on the Continuum of Commodity Characteristics* (Mas-Colell (1975)):

    > The set of *commodity characteristics* [*pure commodities* according to Ostroy and Zame's terminology (1994)] is a compact metric space X; *individual commodity bundles* (*i.c.b.*'s) [*commodity bundles*] are positive (Borel) measures on X. Let $M(X)$ be the space of measures on X, and $M^+(X)$ the cone of positive measures.
    >
    > Let $C(X)$ be the space of continuous real-valued functions on X. Then, $M(X)$ is the dual of $C(X)$ with the sup norm by Riesz Representation Theorem (see e.g. Royden (1968, p.246)). The weak-star topology (w^*-topology) $\mathcal{T}_{w^*}$ of $M(X)$ is the topology of pointwise convergence on continuous functions, and is the weakest topology for which the mapping $(\varphi, \alpha) \to \varphi.\alpha = \int \varphi(x) d\alpha(x)$ is continuous for every $\varphi \in C(X)$.

Let a *nonatomic* $\mu \in M^+(X)$, such that $\operatorname{supp} \mu = X$, i.e., the set of pure commodities is infinite, be a *reference bundle* against which other commodity bundles may be measured. The canonical case is $X = [0, 1]$ and μ the Lebesgue measure.

– Mas-Colell (1975) further specializes X to $X = X^d \cup \{h\}$ with a prerequisite for (at least) one homogeneous good h, and the restriction of $\alpha \in M^+(X)$, denoted as $\alpha|_{X^d}$ to be integer-valued.

– Characterizations of *physical* and *market* "thickness" due to Ostroy and Zame (1994), discussed in some detail in the preceding **5.3.4**, allow for less substitution between commodity bundles than that assumed by Mas-Colell (1975), while retaining the possibility that initial holdings can be widely varied.

A glance at the following summary tables reveals that the attempts at "bidirectional" incorporations much needed for serious investigations of diverse "large-square economies" are sparse and far from complete. I indicated the promising research agenda by meriting specific Conjectures.

Remark 13: At this outset, a close examination of the subsequent tables leads us to establish the following results:

Theorem 15 (Limit Equilibrium Existence with the Mixed Hyperfinite Space of Agents and the Commodity Space ${}^*X \subseteq {}^*\mathbf{R}^+_\infty$): *Let the internal* $\mathcal{E}_\omega : A \to {}^*\mathcal{P}_{mo} \times {}^*X \times {}^*\mathbf{R}_+$ *be a hyperfinite exchange economy constructed from* $\mathcal{E}_{\nu,\omega}$ *in* Theorem 1 *of* Section **2**, *plus the projection of* $\mathcal{E}_\omega$ *onto* ${}^*\mathbf{R}_+$, $\mathrm{proj}_{{}^*\mathbf{R}_+} \mathcal{E}_\omega = \lambda : A \to {}^*\mathbf{R}_+$, *with* $\lambda(a) \gtrsim 0$ *denoting the "weight" of* $a \in A$ *such that* $\sum_{a \in A} \lambda(a) = 1$..

Let A be divided into two groups, $A_1 = \{a_1, a_2, \ \ldots \ , a_m\}$ *and* $A_0 = \{a_{m+1}, a_{m+2}, \ \ldots \ , a_\omega\}$ *for some fixed* $m \in \mathbf{N}$, *independent of* ω. *Agents in* A_1 *will be identified as "large" agents, while those in* A_0 *as "small" agents in the following sense*:

$$(\forall a \in A_1)[\lambda(a) \gtrsim\!\!\!\!/\; 0],$$

and

$$(\forall a \in A_0)[\lambda(A) \simeq 0].$$

Let $p \in \left\{ p \in {}^*\mathbf{R}^+_\infty \middle| \; \|p\|_\infty \le 1,\, p^i \ge \dfrac{1}{\sqrt{\omega}} \ (\forall i \in {}^*\mathbf{N}) \right\}$, $g : A \to {}^*\mathbf{R}_\infty$ *and* $S \subset A$ *be as described in Theorem 1. Denote by* p_c *the nonstandard extension of* ${}^\circ p$, *the standard part of* p, *and write* $p = p_c + p_\infty$.

Define $f : A \to {}^*\mathbf{R}_\infty$ *by*

$$f^i(a) = \begin{cases} g^i(a) + e^i(a) - I^i(a) - \frac{p^i_\infty(I^i(a) - e^i(a) - g^i(a)))}{p^i_c} & \text{for } 1 \le i \le \nu \\ \frac{p.(I(a) - e(a))}{p^{\nu+1}_c} & \text{for } i = \nu + 1 \\ 0 & \text{for } i \ge \nu + 2. \end{cases}$$

Then,

(1) $p_c \in {}^*\ell^+_1$,

(2) *for all* $a \in S$, $f(a) \in d(p_c, a)$,

and

(3) $\sum_{a \in A} f(a)\lambda(a) \lesssim 0$.

Construct from $\mathcal{E}_\omega$ a sequence of finite exchange economies $\{\mathcal{E}_n\}$ by the following procedure:

(i) Choose a finite subset $A_n \subset A$ so that $|A_n| \to \infty$ as $n \to \infty$;

(ii) Assign to each $a \in A$, $^\circ \succ_a$, $^\circ e(a)$ and $^\circ\lambda(a)$, the standard parts of $\succ_a$, $e(a)$ and $\lambda(a)$, respectively.

Theorem 16 (Asymptotic Interpretation of Theorem 15): *Let* $\mathcal{E}_n : A_n \to \mathcal{P}_{mo} \times X \times \mathbf{R}_+$ *be a sequence of finite exchange economies as defined above.*

Let A_n *with* $|A_n| = n \in \mathbf{N}$ *be divided into two groups,* $A_1 = \{a_1, a_2, \ldots, a_m\}$ *and* $A_0 = \{a_{m+1}, a_{m+2}, \ldots, a_n\}$ *for the same* m $(m < n)$ *as specified in the previous* Theorem 15, *independently of the size of the economy* n. *Agents in* A_1 *will be identified as "large" agents, while those in* A_0 *as "small" agents in the following sense:*

$$(\exists \xi \in \mathbf{R}_+)(\forall a \in A_1)[^\circ\lambda(a) \ge \xi],$$

and

$$(\forall \delta \in \mathbf{R}_+)(\exists \nu \in \mathbf{N})[|T_0| > \nu \Longrightarrow (\forall a \in A_0)\,^\circ\lambda(A) \le \delta].$$

Then,

(1) *there exists* $K \subset \mathcal{P}_{mo}$ *compact in* $\mathcal{C}$ *such that* $^\circ \succ_a \in K$ *for all* $a \in A_n$,

and

(2) $\sum_{a\in A_n} {}^{\circ}e(a)\,{}^{\circ}\lambda(a) < \infty$; *and for* $E_n \subset A_n$, $\sum_{a\in E_n} {}^{\circ}\lambda(a) \to 0 \Longrightarrow \sum_{a\in E_n} {}^{\circ}e(a)\,{}^{\circ}\lambda(a) \to 0.$

Moreover, for any $\delta > 0$, *there exists* $\bar{n} \in \mathbf{N}$ *such that, for every* $\mathcal{E}_n$, $n \geq \bar{n}$, *there exist a price* $p_n \in \ell_1^+$ *and a net assignment* $f_n : A_n \to \mathbf{R}_\infty$ *satisfying*

(3) $\sum \{ {}^{\circ}\lambda(a) |\, f_n(a) \in d_n(p_n, a)\} \geq 1 - \delta,$

and

(4) $\sum_{a\in A_n} f_n(a)\,{}^{\circ}\lambda(a) \leq \delta.$

Theorem 17 (Limit Core Equivalence with the Mixed Hyperfinite Space of Agents and the Commodity Space ${}^*X \subseteq {}^*\mathbf{R}_\infty^+$): *Let* $\mathcal{E}_\omega : A \to {}^*\mathcal{P}_{mo} \times {}^*X \times {}^*\mathbf{R}_+$ *be a hyperfinite exchange economy as defined in* Theorem 15 *above.*

Then, given $f \in \mathcal{C}(\mathcal{E}_\omega)$, *there exists* $p \in \left\{ p \in {}^*\mathbf{R}_\infty^+ \middle|\, p.\sum_{a\in A} e(a) = 1 \right\}$ *such that*

(1) $\sum_{a\in A} |p.\,(f(a) - e(a))\,\lambda(a)| \simeq 0,$

(2) $\sum_{a\in A} \Big| \inf \{p.\,(f(a) - e(a))\,\lambda(a)|\; x \succ_a f(a)\} \Big| \simeq 0.$

Theorem 18 (Asymptotic Interpretation of Theorem 17): *Let* $\mathcal{E}_n : A_n \to \mathcal{P}_{mo} \times \mathbf{R}_\infty^+ \times \mathbf{R}_+$ *be a sequence of finite exchange economies as defined in* Theorem 16 *above.*

Then, for any $\delta > 0$, *there exists* $\bar{n} \in \mathbf{N}$ *such that, for every* $\mathcal{E}_n$, $n \geq \bar{n}$, *given* $f \in \mathcal{C}(\mathcal{E}_n)$, *there exist a price* $p_n \in \ell_1^+$ *and a net assignment* $f_n : A_n \to \mathbf{R}_\infty$ *such that*

(1) $\sum_{a\in A_n} \Big| p_n.\,(f_n(a) - {}^{\circ}e(a))\,{}^{\circ}\lambda(a) \Big| \leq \delta,$

(2) $\sum_{a\in A_n} \Big| \inf \{p_n.\,(f_n(a) - {}^{\circ}e(a))\,{}^{\circ}\lambda(a)|\; x \succ_a f(a)\} \Big| \leq \delta.$

In the light of the detailed discussions in **5.3.4** on the market thickness, or the relative size requirement, the following conjectures are straightforward, some of which are under way while others have not been even begun, or else whose precedents were overlooked in my pre-research investigations.

- CONJECTURE (A): Exploit the parallelism between the core equivalence/equilibrium existence with the continuum of agents and those with the mixed measure space of agents. That is, adapt the existence proof, as surveyed in HILDENBRAND (1974, SECTION **2.2**), to get around additional difficulties due to the mixed nature of the space of agents, encountered and solved in the proofs of SHITOVITZ (1973, THEOREMS A-D).

- CONJECTURE (B): Repeat the preceding CONJECTURE (A), this time to generalize the hyperfinite existence theorem, e.g., BROWN and LEWIS (1981a) to KHAN's characterization of mixed hyperfinite space of agents (1976, SECTION **5**).

- CONJECTURE (C): Deduce for $\mathcal{G} = \{\mathcal{E}_n\}$ by an asymptotic interpretation of the result for $\mathcal{E}_\omega$ expected of the previous CONJECTURE (B) in the same manner as KHAN (1976, THEOREMS 1-4) are derived from his THEOREMS L1 - L4.

- CONJECTURE (D): Assume the *Finite Characterization of Mixed Space of Agents* (Adapted from KHAN (1976, SECTION **2**)) in the above. Then, replacing the steps involving the external entities from the proofs based on CONJECTURE (B), and KHAN (1976, SECTION **6**), respectively, and preferably explicating error terms in $\frac{1}{n}$ or $\frac{1}{\sqrt{n}}$, where $n = |A|$.
  - A finite-dimensional special case of THEOREM 19 or 20 of the subsequent CONJECTURE (E)
  - The upper bound may be sharpened so as to be independent of M, by a direct resort to the finite-dimensional SHAPLEY-FOLKMAN THEOREM, instead of its infinite-dimensional generalization, THEOREM 8 in SECTION 4.

- CONJECTURE (E): Repeat the preceding CONJECTURE (D), this time by coping with additional complications due to the enlarged X from $\mathbf{R}^n_+$ to $\mathbf{R}^+_\infty$ in ways similar to the ELEMENTARY EQUILIBRIUM EXISTENCE THEOREM with $\mathbf{R}^+_\infty$ in NOMURA (1993a, THEOREM 1), quoted as THEOREM 4 in SECTION 2, and the ELEMENTARY CORE EQUIVALENCE THEOREM with $\mathbf{R}^+_\infty$ in NOMURA (1992c, THEOREM 1), quoted as THEOREM 6 in SECTION 3, respectively.

THEOREM 19 (Elementary Equilibrium Existence with the Mixed Finite Space of Agents and the Commodity Space $X \subseteq \mathbf{R}^+_\infty$): *Let $\mathcal{E} : A \to \mathcal{P}_{mo} \times X \times \mathbf{R}_+$ be a finite exchange economy.*

Let A with $|A| = n \in \mathbf{N}$ be divided into two groups, $A_1 = \{a_1, a_2, \ \dots \ , a_m\}$ and $A_0 = \{a_{m+1}, a_{m+2}, \ \dots \ , a_n\}$ for the some m $(m < n)$, independent of the size of the economy n. Agents in A_1 will be identified as "large" agents, while those in A_0 as "small" agents in the following sense:

$$(\exists \xi \in \mathbf{R}_+)(\forall a \in A_1)[\lambda(a) \geq \xi],$$

and

$$(\forall \delta \in \mathbf{R}_+)(\exists \nu \in \mathbf{N})[|T_0| > \nu \Longrightarrow (\forall a \in A_0)\lambda(A) \leq \delta].$$

Let M be as specified in THEOREM 4 *in* SECTION 2.

Then,

(1) *there exists $K \subset \mathcal{P}_{mo}$ compact in $\mathcal{C}$ such that $\succ_a \in K$ for all $a \in A$,*

and

(2) $\displaystyle\sum_{a\in A} e(a)\lambda(a) < \infty$*; and for $E \subset A$,* $\displaystyle\sum_{a\in E} \lambda(a) \to 0 \Longrightarrow \sum_{a\in E} e(a)\lambda(a) \to 0.$

Moreover, there exist a price $p \in \Delta = \left\{ p \in \mathbf{R}^+_\infty \,\middle|\, p . \sum_{a\in A} e(a) \leq 1, p \gg 0 \right\}$ *and a net allocation $g(a) \in \mathsf{con}\, d(p, a)$ for all $a \in A$.*

Furthermore, for any such g, there exists a selection f with $f(a) \in d(p, a)$ for every $a \in A$ such that

(3) $\displaystyle\left\| \sum_{a\in A} f(a)\lambda(a) \right\|_2 \leq \left\| \sum_{a\in A} \left(f(a) - g(a)\right) \lambda(a) \right\|_2 \leq \frac{2M}{\sqrt{n}}.$

Theorem 20 (Elementary Core Equivalence with the Mixed Finite Space of Agents and the Commodity Space $X \subseteq \mathbf{R}^{+}_{\infty}$): *Let* $\mathcal{E} : A \to \mathcal{P}_{mo} \times \mathbf{R}^{+}_{\infty} \times \mathbf{R}_{+}$ *be a finite exchange economy as defined in* Theorem 19 *above.*

Let M *be as specified in* Theorem 5 *in* Section 3.

Then, given $f \in \mathcal{C}(\mathcal{E})$, *there exists* $p \in \left\{ p \in \mathbf{R}^{+}_{\infty} \middle| \; p. \sum_{a \in A} e(a) = 1 \right\}$ *such that*

(1) $\displaystyle\sum_{a \in A} \left| p.\, (f(a) - e(a))\, \lambda(a) \right| \leq \frac{2\sqrt{2M}}{n^{\frac{3}{4}}},$

(2) $\displaystyle\sum_{a \in A} \left| \inf \{ p.\, (f(a) - e(a))\, \lambda(a) |\; x \succ_a f(a) \} \right| \leq \frac{2\sqrt{2M}}{n^{\frac{3}{4}}}.$

- Conjecture (F): This combination of the spaces of agents and of commodities provides a promising budding research framework, best fit for serious investigations of imperfect competition with a special emphasis on emergence of the market power exploited by a limited number of “large” agents.

  The Market Thickness, or the Relative Size Requirement in **5.3.4** plays a crucial role in these specific contexts.

- Conjecture (G): A natural candidate for the space of agents to work with is the Loeb measure space of agents.

TABLE 6A

EQUILIBRIUM EXISTENCE THEOREMS WITH MIXED SPACES OF AGENTS AND/OR COMMODITIES

	Continuum of Agents	Mixed Measure Space of Agents	Mixed Hyperfinite Space of Agents	Mixed Asymptotic Space of Agents	Finite Mixed Space of Agents (Elementary)
$X \subseteq \mathbf{R}^n_+$	See e.g. HILDENBRAND (1974, SECTION **2.2**)	(A)	(B)	(C)	CONJECTURE (D): •Special Case of THEOREM 19 below. •Sharpen the upper bound.
$X \subseteq \mathbf{R}^+_\infty$	THEOREM 2 (BEWLEY (1972))		THEOREM 15. •True by transfer of (C) for ${}^*\mathbf{R}^\nu_+$, $\nu \in {}^*\mathbf{N}$. •Extend to ${}^*X \subseteq {}^*\mathbf{R}^+_\infty$ by THEOREM 2 in SECT. **2**.	THEOREM 16. •True by an asymptotic interpreta-ition of the HYPERFINITE LIMIT THEOREM 15.	THEOREM 19. (CONJECTURE (E))
Continuum of Commodity Character-istics, Discrete Case	THEOREM 1 (MAS-COLELL (1975))	(F)			
Continuum of Commodity Characteristics	THEOREM 1 (OSTROY and ZAME (1994))	(F)			
Hyperfinite Space of Commodity Characteristics	(G)		(F)		

TABLE 6B

CORE EQUIVALENCE THEOREMS WITH MIXED SPACES OF AGENTS AND/OR COMMODITIES

	Continuum of Agents	Mixed Measure Space of Agents	Mixed Hyperfinite Space of Agents	Mixed Asymptotic Space of Agents	Finite Mixed Space of Agents (Elementary)
$X \subseteq \mathbf{R}^n_+$	See e.g. HILDENBRAND (1974, SECTION **2.1**)	THEOREMS A-D (SHITOVITZ (1973))	THEOREMS L1-L4 (KHAN (1976))	THEOREMS 1-4 (KHAN (1976))	CONJECTURE (D): •Special Case of THEOREM 20 below. •Sharpen the upper bound.
$X \subseteq \mathbf{R}^+_\infty$	THEOREM 1 (BEWLEY (1973))		THEOREM 17. •True by transfer of THEOREMS 1-4 of KHAN (1976) for ${}^*\mathbf{R}^\nu_+, \nu \in {}^*\mathbf{N}$. •Extend to ${}^*X \subseteq {}^*\mathbf{R}^+_\infty$, by THEOREM 2 in SECT. **2**.	THEOREM 18. •True by an asymptotic interpretation of the HYPERFI-. NITE LIMIT THEOREM 17.	THEOREM 20. (CONJECTURE (E))
Continuum of Commodity Characteristics, Discrete Case	THEOREM 2 (MAS-COLELL (1975))	(F)			
Continuum of Commodity Characteristics	THEOREMS 3 and 4 (OSTROY and ZAME (1994))	(F)			
Hyperfinite Space of Commodity Characteristics	(G)		(F)		

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